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WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. LXVII.

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Solved Questions.

547. (Stephen Watson.) — A uniform circular well, a feet deep, being filled with water, n men are successively employed to empty it: required the depth emptied by each, when an equal share of the work is done by each man. 68

615. (W. Priestley.) — Suppose a person standing in a railway carriage, which moves uniformly at the rate of 25 miles per hour, were to drop a ball to the bottom of the carriage, a height of 5 feet. Find the length of the track described by the ball in its descent. 59

635. (Rev. T. Gaskin.)—(1) In the line AB to find two points E, F, such that $EA \times AF = AC^2$ and $BE \times BF = BD^2$, when AC and BD are given; E and F being also on different sides of A and B respectively. (2) To find E, F when E and F are on the same side of A, but on different sides of B. (3) To find E and F when they lie wholly between A and B. 59

769. (J. W. Elliott.)—A quantity of corn is to be divided amongst n persons, and is calculated to last a certain time, if each person receives a peck every week; but, during the distribution, it is found that one person dies every week, and the corn in consequence lasts *twice* as long as was expected: required the quantity of corn, and the time it lasted. 84

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1207. (N'Importe.)—To determine the position of a rock (R), the angles subtended at it by the distances between three headlands (A, B, C) were observed, viz.,

$$BRC = 150° 2' 42'', \quad CRA = 133° 25' 57'', \quad ARB = 75° 31' 21'';$$

and it was known, from a previous survey, that AB = 8883, BC = 9870, and CA = 10857 yards. Find the distance of the rock from each of the headlands. 63

1913. (The late Rev. R. H. Wright, M.A.)—Find the condition in order that a straight line passing through an angular point of the triangle of reference shall be a normal to the conic whose equation is

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0. \quad \text{..... } 86$$

2141. (The late Professor Wolstenholme, D.Sc.)—The arc of the curve $x = a(2 - 3\cos\theta + \cos 3\theta)$, $y = 3a\sqrt{2}(2\theta - \sin 2\theta)$, from cusp to cusp, is $14a$. 86

2487. (The late Rev. R. H. Wright, M.A.)—If a conic be circumscribed about a triangle ABC, and tangents be drawn at A, B, C, and produced to meet so as to form respectively three triangles having the sides of the triangle ABC for their bases, find forms for the bisectors of the angles of the external triangles, and the equations to their circumscribing circles in trilinear coordinates. 85

2501. (N'Importe.) — Find the equation whose roots are the differences of the roots of the equation $(a, b, c, d, e)(x, 1)^4 = 0$. 118

2916 & 4813. (I. H. Turrell.) — To construct a triangle geometrically, having given the vertical angle, radius of inscribed circle, and the centre of gravity of the triangle on the circumference of the inscribed circle. 97

3223. (W. Siverly.) — Find the greatest cube which can be cut from the solid generated by the revolution of a cycloid around its base. 72

3647. (J. B. Sanders.)—A tube 30 inches long, and closed at one end and open at the other, was caused to descend in the sea with the open end downward, until the enclosed air occupied only one inch of the tube. How far did it descend? 68

3728. (Rev. A. F. Torry, M.A.)—Taking the eccentricity of the earth's orbit round the sun to be $\frac{1}{60}$, and the year to consist of $365\frac{1}{4}$ days, determine the greatest and least angular velocities of the earth about the sun. 72

3761. (W. Siverly.)—One end of a rod, whose length is equal to the diameter of a spherical shell, is passed through a hole in the shell and

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made to touch every point of the interior surface: find the surface described by the other end of the rod. 102

3854. (Professor Sir R. S. Ball, F.R.S.)—From any point perpendiculars are drawn to the generators of the surface $z(x^2+y^2)-2mxy=0$. Show that the feet of the perpendiculars lie on a plane ellipse. 60

4440. (The late Professor Wolstenholme, M.A., Sc.D.)—If $x_1, x_2, \ldots x_n$ be connected by the equation

$$x_1^2+x_2^2+\ldots+x_n^2-x_1x_2-x_2x_3+\ldots-x_{n-1}x_n = 1,$$

show that the maximum and minimum values of $x_1^2+x_2^2+\ldots x_n^2$ will be the values of $\left(1+\cos\frac{r\pi}{n+1}\right)^{-1}$, where r has all integral values from 1 to n. 78

4701. (W. S. B. Woolhouse, F.R.A.S.)—If, within a given closed area, three points be taken at random as the apices of a triangle, show that (1) the average of the square of the area of all such triangles will be reduced to one-third the value, if one of the points be fixed at the centre; also (2), if within a given volume of space four points be taken at random as the apices of a tetrahedron, the average of the square of the volume of all such tetrahedra will be reduced to one-fourth the value, if one point be fixed at the centre; and (3) that this theorem is true when the enclosed area or volume of space is *of any form whatever*. 81

4800. (Professor Cavallin, M.A.) — Find the average area of the triangle cut off by a random line from an equilateral triangle. 93

4815. (Dr. Artemas Martin.) — If four dice be piled up at random on a horizontal plane, find the probability that the pile will not fall down. 52

4824. (Professor Wolstenholme, F.R.S.)—A large area is to be paved with circular discs of black marble which are to touch; the interstices to be filled up with slabs of white marble. Each piece of pavement must be cut from a square slab: if the price of the slabs per foot of area vary as the length of a side, and if the cost of cutting the marble be a shillings per linear foot, and the price of a slab one foot square be b shillings, show that, in order that the area may be paved with least expense, the radius of each disc must be $(2\pi a^{-1})^{\frac{1}{2}}$. 35

4864. (Professor Cavallin.) — Two points P and Q are taken at random on the area of a vertical circle; find the probability that the time of descent for a particle down the straight line PQ, from P to Q, may be less than that from P down the straight line of quickest descent to the circle. 57

4909. (Professor Genese, M.A.) — Find functions f and ϕ of two variables, so that $\alpha = f(xy)$, $\beta = \phi(xy)$, $x = f(\alpha\beta)$, $y = \phi(\alpha\beta)$. ... 101

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5447. (Rev. W. Roberts, M.A.)—The cone $x^2\cot^2\alpha + y^2\cot^2\beta - z^2 = 0$ intersects the sphere $x^2+y^2+z^2-a^2=0$ in a sphero-conic. Show that the equation of the tubular surface, which is the envelope of a sphere of constant radius k, whose centre moves along this sphero-conic, is had by equating to zero the discriminant of the following cubic in λ,

$$4a^2\sin^2\alpha x^2/(P^2+4\lambda a^2\cos^2\alpha) + 4a^2\sin^2\beta y^2/(P^2+4\lambda a^2\cos^2\beta) - z^2/\lambda = 1,$$

where $P = x^2+y^2+z^2+a^2-k^2$. 62

11895. (Professor Mukhopadhyay, M.A.) — Pliny informs us that "Thales determined the cosmical setting of the Pleiades to have happened in his time 25 days after the Autumnal Equinox." Supposing the longitude of the brightest star in that constellation to have been, at the beginning of the year 1760, according to Dr. Bradley, ♉ 26° 38′ 34″, and its latitude 4° 1′ 36″, the precession of the Equinox 50·34″, annually, and the latitude of Miletus 37° 30′, determine the time when Thales lived. 38

12390. (S. Tebay, B.A.)—Find two rational fractions such that their sum shall be equal to the sum of their squares, which is also a square. 72

12866. (C. Ewing.)—Find three cube numbers whose sum will be both a cube and a square. 120

13049. (Professor Anthony.)—A vertical slit is made in the middle of the side of a rectangular box containing water. Find the time required to empty the box. 54

13142. (Professor Schwatt.)—Prove that, if through the centre of perspective D of a given triangle ABC and its Brocard-triangle A′B′C′ be drawn straight lines so as to pass through S_a, S_b, S_c (S_a, S_b, S_c are the middle points of the sides BC, AC, and AB of the triangle ABC), and if S_aD_1 is made equal to DS_a, S_bD_2 equal to DS_b, S_cD_3 equal to DS_c, then (1) the figures $D_1O'AO$, $D_2O'BO$, and D_3O CO are parallelograms (O and O′ are Brocard-points), (2) the triangles $D_1D_2D_3$ and ABC are equal, and (3) D_1A, D_2B, and D_3C intersect in S (S is the middle point of OO′). 42

13152. (Rev. T. Wiggins, B.A.)—Given a conic with focus S, and P, Q any two points on it ; if the sum of SP, SQ is constant, find the ocus of the middle point of PQ. 33

13167. (Professor de Longchamps.)—Par le centre d'une ellipse E, on fait passer une circonférence Δ. Soient A, B, C trois des points communs aux courbes E, Δ; on suppose que les normales en ces points concourent en P. Trouver (1) le lieu décrit par le centre de Δ; (2) le lieu décrit par le point P. Ces lieux sont des ellipses. 38

13169. (Professor Orchard, M.A., B.Sc.)—Find the length of the evolute of the curve $p = r\sin\gamma$. 69

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13173. (W. J. Dobbs, M.A.)—A circle drawn touching AB at B and and passing through I the in-centre of the triangle ABC, meets AC in H and K. Prove that IC bisects the angle HIK. 41

13177. (M. Brierley.)—From a given point draw two tangents to a given circle and join the points of contact. From the centre of the circle draw any radius (produced if necesary), perpendicular to which draw a line from the given point, and produce it indefinitely. If then lines be drawn from the points of contact of the tangents to intersect in the extremity of the radius before drawn, they will intercept on the indefinite line a portion of it of a constant magnitude. 45

13184. (P. W. Flood.) — In a circle 100 yards in diameter three equal circles are inscribed touching each other and the outer circle. It is required to find the diameter of a circle inscribed in the space at the centre. 67

13190. (R. W. D. Christie.) — Let 1/P, where P is any integer ending in the digits 1, 3, 7, or 9, have $r_1, r_2, \ldots r_p$, remainders; then

$$r^p - 1 = \mathrm{P.Q}, \quad e.g., \quad 24^7 - 1 = 239\,(\mathrm{M}). \ldots\ldots 40$$

13192. (Professor Morley.)—A. can dig a field of potatoes in one hour, and pick the roots from the plants in two hours. B. can dig that field in three hours and pick it in four. C. can dig in five hours and pick in six. What is the least time in which all working together can dig and pick the field? Is a general formula known or obtainable for the least time in which m men can do n things, where no two things can be profitably done simultaneously by the same man? 80

13197. (Professor Finkel.) — Two points are taken at random on the circumference of a semicircle. Find the chance that their ordinates fall on either side of a point taken at random on the diameter. 35

13198. (Professor Sanjána.) — Prove that the average area of a triangle on a given base (b), and having a given vertical angle (α), is $\frac{1}{4}b^2\left(\frac{1}{\tan\alpha}+\frac{1}{\pi-\alpha}\right)$. Prove also that the average area of a triangle inscribed in a circle of given radius (a) with its base in a fixed direction is $\frac{a^2}{\pi}\int_0^\pi \frac{d\theta}{\theta}\sin^2\theta$. Will this expression give the average area of any triangle inscribed in the circle? 82

13205. (Rev. J. Cullen, B.A.) — If ρ be any vector to a curve in space, show that $\int (\mathrm{T}\rho)^n \mathrm{T}d\rho$ (the integral being taken between finite limits) is a maximum or minimum when $(\mathrm{T}\rho)^2\rho'' = n\rho'\mathrm{V}\rho\rho'$, s being the independent variable. Interpret this equation and its first integral. 85

13206. (R. Chartres.) — If a number a be divided at random into

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n parts, find the mean value of the mth power of one of the parts, with its geometrical application. 33

13210. (F. A. L. Kitchin, B.A.) — Two stars whose elements are unknown rise together, and bear S. $\alpha°$ W. at the same moment. Show that the latitude (l) of the place is given by $\tan l = \sin \alpha$. 44

13213. (J. J. Barniville, B.A.)—Prove that

$$\sqrt{5} = \left(\frac{2}{1.1}\right)^2 - \left(\frac{3}{1.2}\right)^2 + \left(\frac{5}{2.3}\right)^2 - \left(\frac{8}{3.5}\right)^2 \quad \ldots\ldots\ldots (S_1),$$

$$= 2 + \left(\frac{1}{1.2}\right)^2 - \left(\frac{1}{2.3}\right)^2 + \left(\frac{2}{3.5}\right)^2 \quad \ldots\ldots\ldots\ldots (S_2).$$

..... 39

13219. (R. W. D. Christie.)—Prove that the sum of the angles in a right-angled primitive spherical triangle is less than three right angles. 31

13225. (Professor Sanjána, M.A.)—From points on a fixed straight line tangents are drawn to a given conic, and triangles are formed by them with their chords of contact. Prove that the locus of the median points (centroids) of the triangles is a parabola when the conic is parabolic, and a cubic curve when the conic is central. Also, trace the curve $3bx - b^2 = 2a^2x^2/(x^2+y^2)$, which is the locus when the conic is a circle. [This cubic is the inverse of an ellipse, and is frequently met with.]... 39

13230. (Professor Cazamian.)—Soient O le centre d'inversion, A et B deux points d'une figure, A′ et B′ les points correspondants de la figure inverse. Démontrer que la médiane et la symédiane du triangle AOB, issues de O, sont respectivement symédiane et médiane du triangle A′OB′. 46

13236. (H. W. Curjel, M.A.)—Prove that the diameters conjugate to a common diameter of two confocal conics are at right angles, and the um of their squares is zero. 71

13239. (J. F. Wilkinson.) — If x, y be the rectangular coordinates of a point and r its radius vector, prove that on each of a series of parabolas an indefinite number of points may be found giving integral values of x, y, r. Establish formulæ for values of x, y, r, in terms of an independent variable susceptible of integral values only. 70

13241. (J. J. Barniville, B.A.)—Prove that

$$\frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{5+1} + \frac{1}{13+1} + \ldots = \frac{\sqrt{5}}{2},$$

$$\frac{1}{1+i} + \frac{1}{3+i} + \frac{1}{i+8} + \ldots = 1 - \frac{i}{2}(\sqrt{5}-1). \ldots\ldots\ldots 100$$

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13244. (P. W. Flood.)—Given the hypotenuse of a right-angled triangle, to construct it, when the rectangle under the segments of one of the other sides made by the bisector of the angle opposite is equal to a given square or a maximum. 42

13246. (Rev. T. Wiggins, B.A.)—A movable line through the origin meets the fixed straight line $lx+my+n=0$ at P, and the fixed conic $ax^2+2hxy+by^2+2qx+2fy+c=0$ at Q and S. R is taken so that P and R divide QS harmonically. Prove that the locus of R may be written in the form $n(ax^2+2hxy+by^2+2qx+2fy+c)=(lx+my+n)(gx+fy+c)$. 43

13248. (M. Brierley.)—A straight line AB and a point P are given in position; through P another line is drawn to intersect AB in B, and PB is so divided in C that PC, PB is given in magnitude and a constant quantity; required the locus of C. 46

13251. (Professor Ignacio Beyens.)—A uniform rigid rod, of length $2a$, can turn in a horizontal plane about its middle point. At one end a string is tied which passes over a fixed pulley, vertically over that end, and at a distance b from it, and is then fastened to a given weight. The rod is then turned through an angle θ, and kept at rest in that position by a horizontal force P perpendicular to the rod through its other end. Prove that P will be a maximum if $\tan^4\frac{1}{2}\theta = b^2/(b^2+4a^2)$, where a is the radius of the circle circumscribing the polygon, r the distance of P from its centre O, and θ the angle OP makes with the radius drawn to any vertex of the polygon. 54

13258. (Professor Siverly.)—Of all cones which develop an ellipsoid and have their bases in the tangent plane at a given point P, and are of the same altitude, that is the least which has its vertex in the diameter through P; and of all which have their vertices in this diameter, that is the least whose axis is twice this diameter. 37

13266. (J. J. Barniville, B.A.)—Sum the series

$$\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\frac{8}{1+x^8}+\dots \quad \text{..... 53}$$

13267. (R. Chartres.)—If a point P be taken at random in a triangle, and lines PQ, PR be drawn parallel to the sides, prove that the mean value of $QR=\frac{1}{3}a$, and the mean value of $QR^2=\frac{1}{6}a^2$, or the mean value of $PQR=\frac{1}{9}\Delta$; also, if three points P, Q, R be taken at random in a triangle ABC, and DE, DF, EF be drawn through them parallel to the sides, then the mean value of $PQR=\frac{1}{4}DEF$, of $DEF=\frac{1}{3}\Delta$; therefore of $PQR=\frac{1}{12}\Delta$. 33

13270. (J. W. West.)—Evaluate dy/dx on the supposition that $\int ds$ is an assigned magnitude, when $\int y\,dx$ (immersed in a fluid) gives a maximum moment of total pressure. 53

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13271. (R. Knowles, B.A.)—If p_r denote the coefficient of x_r in the expansion of $(1+x)^{-\frac{1}{2}}$, prove that

$$p_1-\tfrac{1}{2}p_2+\tfrac{1}{3}p_3-\text{\&c. ad infin.} \equiv 2\log\tfrac{1}{2}(1+2^{\frac{1}{2}}). \ldots\ldots\ldots 52$$

13272. (P. W. Flood.)—Given two chords of a circle, the angle at which they meet, and one of the external segments, describe the circle. 91

13277. (Professor Cochez.) — Limite de $y = (x+\cos x+\tan x)^{1/(\sin x)}$ quand x tend vers zéro. 70

13283. (Professor Aley.) — Telegraph poles are a yards apart; for how many minutes must one count poles in order that the number of poles counted may be equal to the number of miles per hour that the train is running? 45

13284. (Professor Dickson.) — Find the general term in the series 1, 3, 10, 35, 126, 462, 1716, 6435, 24310, ... which plays a remarkable part in some recent theorems in the theory of regular polygons. 108

13287. (Editor.)—Prove that, in any triangle, we have, using the ordinary notation, $(r_a-r)(r_b-r)(r_c-r) = 4Rr^2$. 45

13289. (D. Biddle.) — A sphere of plastic material is deformed by gradual flattening underneath, but so that the curved surface maintains sphericity of ever-increasing radius around a fixed centre, until a hemisphere results. Find the solid contents of the figure traced by the horizontal plane surface. 69

13290. (Rev. J. Cullen, B.A.)—One end of a heavy beam lying in a horizontal plane is fastened by a hinge to the bottom of an upright post. To the other end of the beam is attached a light rope passing over a pulley at the top of the post. Find the curve in which a weight fastened to the rope must slide in order that the system may be in equilibrium for all positions of the beam; neglecting friction and assuming the weight and beam to move in the same vertical plane. 30

13291. (H. W. Curjel, M.A. Suggested by Quest. 13258.) — If $x = 0$ is the base of a cone, vertex P, which envelopes a sphere $x^2+y^2+z^2 = 2ax$, show that locus of P (1) when the area of the base of the cone is constant is $(x^2+y^2+z^2-2ax)\,x = m(x-2a)^3$; (2) when the volume of the cone is constant is $(x^2+y^2+z^2-2ax)\,x^3 = n(x-2a)^3$. ... 55

13293. (W. J. Dobbs, M.A.)—Four heavy uniform rods of the same thickness and material, and of lengths a, b, c, d, are freely jointed at their extremities to form a quadrilateral framework. A fine string of given length connects the hinges ab and cd. The framework rests in a vertical plane with the rod d held in a given position. Prove the following method for determining the tension of the string. Draw TBC vertically downwards making $TB = \frac{1}{2}(a+b)$ and $BC = \frac{1}{2}(b+c)$. Let the straight

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lines through B, C parallel to the rods b, c respectively meet in O. Draw TA parallel to the string and OA parallel to the rod a to meet TA in A. Then AT represents the length of a rod whose weight is equal to the tension of the string. 56

13294. (C. Bickerdike.) — Assuming that Venus and the Earth move in circular orbits with the Sun at the centre, what is the angle of position of the former when at its greatest brilliancy, the intensity of light varying inversely as the square of the distance? 55

13295. (I. Arnold.)—If two weights P and Q sustain each other on two inclined planes AB, AC, having a common altitude A, and if a circle be described through the points A, P, Q, and P and Q be connected by a string passing over a pulley E in any part of this circumference, prove that P and Q will still be in equilibrium. 90

13297. (Rev. T. Wood Robinson, R.N.) — Prove that a rhumb line may be projected into an equiangular spiral; find its equation and deduce the formula $\tan \text{course} = \frac{\text{diff. long.}}{\text{mer. diff. lat.}}$. 75

13301. (R. Chartres.)—If n points be taken at random, one in each side of a regular polygon of n sides, prove that the mean value of the inscribed figure $= \cos^2(\pi/n)$, if the area of the polygon be unity. ... 51

13302. (R. W. D. Christie.)—Prove that the tangents and cotangents of the half angles of any triangle are the roots of the equations

$$sx^3-(4R+r)x^2+sx-r=0, \quad rx^3-sx^2+(4R+r)x-s=0,$$

where the letters have the usual meanings. 41

13303. (Professor Neuberg.) — Sur les côtés BC, CA, AB d'un triangle ABC, on marque les sommets A′, B′, C′ d'un triangle A′B′C′ dont les côtés a', b', c' vérifient la relation donnée $\alpha a'^2+\beta b'^2+\gamma c'^2=0$. ($\alpha$) Si l'on donne le point C′, la droite A′B′ enveloppe une courbe de la quatrième classe. (β) Si l'on donne les points A′, B′, le point C′ occupe sur AB deux positions C'_1, C'_2. Les points C'_1, C'_2 coïncident, lorsque la droite A′B′ enveloppe une certaine courbe de la quatrième classe. (γ) Enfin, si les points A′, B′, C′ sont sur une même droite, cette droite enveloppe une certaine courbe de la quatrième classe. 28

13306. (Professor Sanjána.) — In an isosceles triangle ABC, right-angled at C, CA is bisected in D, and DG is drawn perpendicular to AB; prove that AG = ⅓GB. Hence, or otherwise, inscribe (geometrically) in a quadrant of a circle a square having two angular points on the arc. [The first part proves that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{1}{4}\pi$.] 47

13307. (Professor Crofton, F.R.S.) — If m things are distributed among $n+x$ persons, in such a way that a particular set of n among them

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must each receive something, prove that the number of possible distributions is $N = \Delta^n x^m$. 25

13311. (Professor Schwatt.) — The Simson-line belonging to one point of intersection of Brocard's diameter of a triangle with the circumcircle of this triangle is either parallel or perpendicular to the bisector of the angle formed by the side BC of the triangle ABC and the corresponding side B'C' of Brocard's triangle. 32, 73

13312. (Professor Genese.)—Let N be any integer, a, b, c, ..., the prime numbers less than $\sqrt{N}$, and

$$\phi(N) = \{1 - \Sigma(1/a) + \Sigma(1/ab) - \Sigma(1/abc) + \ldots) N\}$$

in every operation of which, such as N/a, N/b, N/ab, only the integral part of the quotient is taken. Then, in general, $\phi(N)$ + the number of the primes a, b, c ... = the number of primes (including unity) less than N; but, if N be the square of a prime number, the result of the count must be diminished by 1. 27

13314. (Professor Matz.) — Find the average area of the random sector whose vertex is a random point in a given circle..... 28

13315. (Professor Finkel.)—Find the average length of a line drawn at random across the opposite sides of a rectangle whose length is l, and breadth b. 68

13316. (Professor Wilkes.)—Let n be any number, and let $n^3 + 1 = x$; prove that $x^3 + (2x-3)^3 + (nx-3n)^3 = n^3x^3$. 55

13318. (Professor Barisien.) — Dans un triangle ABC on considère les centres O_B et O_C des cercles ex-inscrits dans les angles B et C et on projette ces centres en O'_B et O'_C sur le côté BC. La droite O_CO_B rencontre BC en A'. Montrer que (1) les points A et A' sont conjugués harmoniques des points O_C et O_B; et (2) l'ellipse ayant pour sommets du grand axe les points O'_C et O'_B et passant par A jouit de la propriété d'être tangente à la droite O_CO_B et d'avoir pour foyers les points B et C. 43

13321. (D. Biddle.)—Standing within a circular cinder-track, whilst A. and B. cycle round it, I observe that each is going at a uniform rate, but that B., who has been handicapped, is gaining on A., their relative speeds being 10 : 9. As A. crosses the further extremity of the diameter on which I am standing, the arc between him and B. subtends, from my point of view, an angle of 40°. I then turn my back on the centre, and, as A. crosses the other extremity of the diameter, I observe that the corresponding angle between the riders is 50°. Find (1) my position on the said diameter, (2) the point at which B. will catch up A., their speeds remaining the same. 29

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13322. (Rev. J. Cullen, B.A.)—Prove that

$$\int_0^K \mathrm{cn}^2 u\,(\mathrm{cn}^2 u\,\mathrm{dn}^4 u + 2k'^2\,\mathrm{sn}^2 u)\,du \,/\, \mathrm{dn}^4 u = \left\{2(1+k^2)\,\mathrm{E} - 2\mathrm{K} - k^2\mathrm{K} + 3k^4\mathrm{K}\right\}/3k^4. \quad 76$$

13323. (Rev. T. Wiggins, B.A.)—Given, as an identity,

$$\int_0^{\frac{1}{2}\pi} (a+\sin^2 x)^n \cos x\,dx = \int_0^{\frac{1}{2}\pi} (1+a-\sin^2 x)^n \sin x\,dx,$$ prove that

$$a^n + \frac{n}{3}a^{n-1} + \frac{n(n-1)}{2.5}a^{n-2} - \frac{n(n-1)(n-2)}{3!.7}a^{n-3} + \ldots$$

$$= (a+1)^n - \frac{2n}{3}(a+1)^{n-1} + \frac{2^2 n(n-1)}{3.5}(a+1)^{n-2}$$

$$- \frac{2^3 n(n-1)(n-2)}{3.5.7}(a+1)^{n-3} + \ldots.$$

..... 71

13324. (W. J. Dobbs, M.A.) — Four fine light rods are smoothly jointed at their extremities to form a quadrilateral which can be inscribed in a circle. The opposite joints are connected by two fine light strings in a state of tension. Prove that the thrusts in the rods and the tensions of the strings are proportional to the opposite sides and diagonals of the quadrilateral, respectively. 32

13326. (R. F. Davis, M.A.)—If A, B, C be the angles of a triangle, prove they satisfy a relation of the form $\cos^2\theta + \lambda\cot\theta = \mathrm{P}$. 37

13327. (R. Knowles, B.A.)—PQ, PR are chords of a parabola at right angles; prove that (1) the locus of the pole of QR is a right line; (2) as P varies, the envelope of this locus is an equal parabola. 30

13328. (J. J. Walker, F.R.S.)—Show that the line of ultimate intersection of the planes $ax + by + cz = 1$, where the parameters a, b, c are connected by the relation $1 + \Sigma a + \Sigma bc = 0$, is a generator of the quadric $1 + \Sigma x + \Sigma yz = 0$. 26

13329. (G. Heppel, M.A.) — Find the least possible value of the acute angle between the diagonals of a quadrilateral described about a circle. 56

13330. (W. C. Stanham.)—Prove, geometrically, that the locus of the foci of parabolas which touch a central conic and its principal axes is the inverse of the evolute of the conic, with respect to the concentric circle through its foci. 91

13331. (Rev. W. A. Whitworth, M.A.)—If n chords be drawn at

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random in a circle, show that the chances of

$\frac{1}{2}n(n-1)$ intersections $= N$, $\frac{1}{2}n(n-1)-1$ intersections $= nN$,

no intersections $= 2^n \div (n+1)!$, and one $= 2^n n(n-1) \div (n+2)!$;

where $N = 2^n n!/(2n)! = 1 \div$ product of the first n odd numbers. ... 48

13332. (G. N. Bates, B.A.) — Prove the following method for inscribing a pentagon in a circle:—Let AFC be a circle, AF a diameter. Divide AF into five equal parts; with A and F as centres and radius AF, describe arcs of circles cutting in K. Join K with H, the second point of division; produce KH to cut circle in G. AG is the side of the pentagon.—Show that, in a circle of radius 1″, the side first obtained is of length 1·174...″. (True length should be 1·1755...″.) 61

13333. (J. J. Barniville, B.A.)—Prove that

$$\frac{1}{1.2.3}+\frac{1}{1.3.5}+\frac{1}{2.5.8}+\frac{1}{3.8.13}+\ldots=\tfrac{1}{4},$$

$$\frac{1}{1.2.3}-\frac{1}{1.3.5}+\frac{1}{2.5.8}-\ldots=-\tfrac{3}{4}+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\ldots. \quad \ldots\ldots 36$$

13335. (Professor Beyens.)—Obtain a solution of $dy/dx + (ax+b)y = c$, where a, b, and c are constants; y to be expressed in terms of known functions of x, or as a rapidly convergent series of integral powers of x. 70

13337. (P. W. Flood.) — Euler says it would be difficult to find x so as to make $p^4 - 2p^2x^2 + x^4 + x$ a square; but show that it is not difficult. 64

13338. (Rev. T. T. Roach, M.A.)—A man riding from one point to another has to cross, first, a heavy sand on which he can only ride $6\frac{1}{2}$ miles per hour, and then firm grass on which he can ride 10 miles per hour. Find the shortest time in which he can ride, given that the perpendicular distances of the two points from a straight road separating the two districts are 4 and 5 miles, respectively, and that the feet of the perpendiculars are 15 miles apart. 27

13340. (C. E. Hillyer.)—O is a fixed point, P any point in a given straight line, and PK is drawn equally inclined with PO to a fixed straight line through O. Prove that PK always touches a fixed parabola; and, further, that if PK′ be drawn equally inclined with PO to a second fixed straight line through O, then each side of the triangle PKK′ touches one of three fixed confocal parabolas. 58

13341. (R. Chartres.)—If three points be taken at random in the sides of a triangle, one in each side, prove that the mean value of each $= \frac{1}{4}\Delta$; also, if two points be taken at random in the sides, the mean value of the triangle AFE cut off $= \frac{1}{4}\Delta$. 101

13342. (Professor Neuberg.)—On donne un losange ABCD, et l'on

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joint un point quelconque P de la diagonale AC aux sommets B et D, puis un point quelconque Q de la diagonale BD aux sommets A et C. Démontrer que les quatre droites ainsi menées forment un quadrilatère inscriptible, et trouver le lieu du centre de la circonférence circonscrite à ce quadrilatère lorsque P est fixe et que Q se meut sur BD. 96

13345. (Professor Hudson.)—Find the equation of an ellipsoid referred to the planes of its central circular sections and a central plane at right angles to them. If these are rectangular axes, prove that the squares of the axes are in harmonical progression, and that the equation takes the form $\frac{(x+z)^2+y^2}{c^2} + \frac{(x-z)^2+y^2}{a^2} = 2$. 87

13347. (Professor Steggall.)—A number n of given forces act at a point O: the resultant, by the polygon construction, is OP. Show that in the construction of *all* the $n!$ polygons by which this resultant can be obtained there are $n \cdot 2^{n-1}$ straight lines. 117

13348. (Professor H. L. Orchard, M.A., B.Sc.) — Sum to n terms the series $s = 3+5+14+36+88+208+$ &c. 100

13351. (Professor Krishnamacharry.) — If a weight P balance a weight W in that system of pulleys in which each pulley hangs by a separate string, show that, if P be changed to P′ and W to W′, P′ will descend with acceleration f such that

$$f\{2^{2n}P' + W' + \tfrac{1}{2}(2^n+1)(2^nP-W)\} = 2^n g\{2^n(P'-P) + W - W'\},$$

all the pulleys being of equal weight. 46

13356. (Professor Percépied.)—Par le centre d'un hexagone régulier ABCDEF, on mène une sécante quelconque qui coupe en G, H les diagonales AC, AE. Trouver le lieu du point d'intersection des droites BG et FH. 79

13358. (H. B. Billups, B.A.)—P is the orthocentre of a triangle; show that PA, −PB, −PC are the roots of the equation

$$x^3 + 2(r_a - R)x^2 + \{r_a^2 - 4R^2 + (s-a)^2\}x + 2R\{(2R-r_a)^2 - (s-a)^2\} = 0.$$
..... 65

13359. (C. E. Hillyer.)—If the polar of a point O with respect to a conic meet any pair of conjugate diameters in F and f, and EOe be drawn antiparallel to Ff with respect to the angle FCf meeting CF in E, and CF in e, prove that F, E and f, e are pairs of inverse points with respect to the director circle; hence show (1) that the circumcircle of a self-conjugate triangle is orthogonal to the director circle; (2) by taking O on the curve, that tangents at right angles intersect on the director circle. Examine the special cases of the parabola and rectangular hyperbola. 74

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13361. (W. C. Stanham.) — Construct (a) a right-angled triangle having its angular points on the given rectangular hyperbola $x^2-y^2=a^2$ (rectangular coordinates), and its centroid at the point $(\frac{1}{3}a\cos\phi, \frac{2}{3}a\tan\phi)$, ϕ being any given angle; and show (b) that, both in this case and in the case in which the centroid is to be at the point $(\frac{2}{3}a\sec\phi, \frac{1}{3}a\cot\phi)$, the determination of the coordinates of the angular points of one such triangle will not necessitate the solution of an equation of a degree higher than the second. 64

13362. (Rev. T. Roach, M.A.)—If $\Sigma\cos^4\frac{1}{7}\pi$ denote

$$\cos^4\tfrac{1}{7}\pi+\cos^4\tfrac{2}{7}\pi+\cos^4\tfrac{3}{7}\pi,$$

find the value of $\Sigma\cos^4\frac{1}{7}\pi$, $\Sigma\sin^4\frac{1}{7}\pi$, $\Sigma\sec^4\frac{1}{7}\pi$;

$\Sigma\cos^6\frac{1}{7}\pi$, $\Sigma\sin^6\frac{1}{7}\pi$, $\Sigma\tan^6\frac{1}{7}\pi$, $\Sigma\sec^6\frac{1}{7}\pi$;

and $\Sigma\cos^8\frac{1}{7}\pi$, $\Sigma\sin^8\frac{1}{7}\pi$, $\Sigma\tan^8\frac{1}{7}\pi$, $\Sigma\sec^8\frac{1}{7}\pi$. 87

13368. (W. J. Dobbs, M.A.)—A, B, H, K are four fixed points, and L a fixed straight line. Any point X is taken in HK, and AX, BX meet the line L in a, b respectively; Ha, Kb intersect in Y. Prove that, as X traces out the line HK, Y traces out a straight line which passes through the intersection of AB, ab. 75

13369. (Rev. J. Cullen, B.A.)—Solve the partial differential equation

$$z\left\{\left(\frac{\partial z}{\partial c}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2\right\}=x-y. \quad \text{..... 120}$$

13371. (R. W. D. Christie.) — If G, O, a, b, c are the centroid, circumcentre, and sides of any triangle, prove that

$$(1)\quad 36\mathrm{GO}^2=9a^2\cot^2\mathrm{A}+5a^2-4b^2-4c^2;$$

and (2), if the triangle be right-angled, GO = R/3. 76

13372. (R. Knowles, B.A.) — If PP′ be a diameter of a central conic, and a tangent from P meets a chord P′R in Q, prove that the tangent from R bisects PQ. 100

13373. (P. W. Flood.)—From a given point in the diameter produced of a given semicircle, draw a line cutting the circumference in two points such that, if perpendiculars be drawn from them on the diameter, their rectangle will be given. 118

13375. (Professor Sylvester, Athenæum Club, 18*th Dec.*, 1896.)—If

$$\frac{\mathrm{F}x}{(1-x)^n}=1+2^{n-1}x+3^{n-1}x^2+\&c.,$$

prove that Fx is a symmetrical function of the degree $(n-2)$ in x, the sum of whose coefficients is the factorial of $(n-1)$.

Ex.—Let $n=5$. Call $\mathrm{F}x=1+\mathrm{B}x+\mathrm{C}x^2+\mathrm{D}x^3$; by the rule given

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above, D = 1, C = B, and 1 + B = 12, so that B = 11, C = 11, D = 1, and accordingly

$$\frac{1+11x+11x^2+x^3}{(1-x)^5} = 1+16x+81x^2+256x^3+\dots .$$

Let $n = 6$. Call $Fx = 1+Bx+Cx^2+Dx^3+Ex^4$. Then obviously B = 32 − 6 = 26, and, by our rule, D = 26, E = 1, and C = 120 − 54 = 66. So, if $n = 7$, if $Fx = 1+Bx+Cx^2+Cx^3+Bx^4+x^5$, we have B = 64 − 7 = 57, C = 1½² − 58 = 302, and it will be found that

$$\frac{1+57x+302x^2+302x^3+57x^4+x^5}{(1-x)^7} = 1+64x^2+729x^3+4096x^4+\dots .$$

Thus, with scarcely any work of computation, we may find the following table of coefficients in Fx corresponding to all values of n from 1 to 9 inclusive, viz.: 1; 1; 1, 1; 1, 4, 1; 1, 11, 11, 1; 1, 26, 66, 26, 1; 1, 57, 302, 302, 57, 1; 1, 120, 1312, 2416, 1312, 120, 1; 1, 247, 3693, 16219, 16219, 3693, 247, 1. 57

13385. (H. B. Billups, B.A.) — Prove that (1) a, $-b$, $-c$ are the roots of the equation

$$x^3+2(s-a)x^2+\{(s-a)^2+r_a^2-4Rr_a\}x-4Rr_a(-a) = 0;$$

and (2) cos A, −cos B, −cos C are the roots of

$$4R^2x^3+4R(r_a-R)x^2+\{(s-a)^2+r_a^2-4R^2\}x-(s-a)^2+(2R-r_a)^2 = 0,$$

so that the triangle is acute-angled or obtuse-angled according as $s-a$ is numerically greater or less than $2R-r_a$. 65

13388. (W. J. Greenstreet, M.A.) — A conic cuts the sides of a triangle ABC in D, D′; E, E′; F, F′, respectively. The tangents at D, D′ meet AB, AC in K, K′; L is the fourth harmonic of B with respect to F, F′; M that of C with respect to E, E′. Show D′L, DM, KK′ are concurrent. 107

13390. (D. Biddle.)—Construct the triangle of which AD, BE, CF are the unknown perpendiculars, having given the three products BC . CD, CA . AE, AB . BF. 77

13392. (P. W. Flood.)—From a given point in the diameter produced of a given semicircle, draw a line cutting the circumference in two points such that, if perpendiculars be drawn from them on the diameter, their rectangle will be given. 62

13394. (H. Orfeur.)—Besides the circle, there is another curve possessing the property radius of curvature = radius vector. How may it be analytically described? Show that it is a particular involute of the involute of a circle. 110

13399. (Rev. J. Cullen, B.A.)—Find the general term and the sum to n terms of the series

$$1+2+16+77+391+1952+\dots . \quad \dots\dots 95$$

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13401. (Professor Neuberg.) — Les côtés AB, BC, CD, DA d'un quadrilatère ABCD touchent une circonférence aux points a, b, c, d.

Démontrer que $\frac{(ac)^2}{(bd)^2} = \frac{AD.BC}{AB.DC}$. 79

13402. (Professor C. Cochez.) — On donne deux axes rectangulaires Ox, Oy. Une parabole reste tangente à Oy en O et passe par un point A donné sur Ox, ayant pour abscisse $2a$. Trouver et construire (1) le lieu du point de contact d'une tangente à la parabole, parallèle à Ox; (2) l'enveloppe de la directrice de la parabole; (3) le lieu du foyer de la parabole; (4) le lieu du sommet de la même parabole. 106

13405. (Professor Ramaswami Aiyar.)—If a system of similar ellipses have a common centre and all of them touch one of them, the locus of their foci consists of two circles. 83

13408. (Professor Chakrivarti.)—A conic always touches four given straight lines; prove that the chord of intersection of the circle, described about any one of the triangles formed by three of these straight lines, with the circle which is the locus of the intersection of two tangents to the conic at right angles to each other, always passes through a fixed point. 88

13409. (Professor Sarkar.)—Three circles, touching each other externally, are all touched by a fourth circle including them all. If a, b, c be the radii of the three internal circles, and α, β, γ the distances of their centres from that of the external circle, respectively, prove that

$$2(\beta\gamma/bc + \gamma\alpha/ca + \alpha\beta/ab) = 4 + \alpha^2/a^2 + \beta^2/b^2 + \gamma^2/c^2. \text{ ..... } 112$$

13413. (Professor Genese, M.A.)—The late Professor Catalan wrote that he had during thirteen years, *à diverses reprises*, unsuccessfully tried to prove directly the following theorem, viz., a, b, c,... k being any unequal quantities,

$$\left\{\frac{1}{a-b} + \frac{1}{a-c} + \frac{1}{a-d} + \ldots + \frac{1}{a-k}\right\} \frac{1}{(a-b)^2(a-c)^2\ldots(a-k)^2}$$
$$+ \left\{\frac{1}{b-a} + \frac{1}{b-c} + \ldots + \frac{1}{b-k}\right\} \frac{1}{(b-a)^2(b-c)^2\ldots(b-k)^2} + \ldots$$
$$+ \left\{\frac{1}{k-a} + \frac{1}{k-b} + \frac{1}{k-c} + \ldots \frac{1}{(k-h)}\right\} \frac{1}{(k-a)^2(k-b)^2\ldots(k-h)^2} \equiv 0:$$

show that the identity may be established either without, or, better, with, the aid of the differential calculus. 83

13420. (Capt. H. Brocard. Généralisation de la Quest. 8407.)—Les droites qui joignent les sommets d'un triangle ABC à un point Δ déterminent sur les côtés des points α, β, γ, par chacun desquels on mène

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des parallèles aux deux autres côtés du triangle donné. L'on obtient ainsi les points (λ', λ), (μ', μ), (ν', ν) sur les côtés a, b, c, en suivant les côtés du triangle respectivement dans l'ordre direct des côtés et dans l'ordre inverse. Cela posé, les triangles $\lambda'\mu'\nu'$, $\lambda\mu\nu$, ainsi déterminés, sont homologiques avec le triangle ABC, et admettent pour centres d'homologie deux points δ' et δ, appelés aussi *point direct* et *point rétrograde* par rapport au point Δ (E. Lemoine). Si le point initial Δ est le point de Lemoine K, δ' et δ deviennent les points de Brocard, ω' et ω. M. de Longchamps a établi que, dans ce cas, les trois triangles $\lambda'\mu'\nu'$, $\lambda\mu\nu$, $\alpha\beta\gamma$ sont équivalents. Démontrer qu'il en est encore ainsi dans le cas général défini par la construction précédente. 89

13422. (A. Gordon.)—Show geometrically how to find a point M on an ellipse such that, P, Q being two points on the same side of any diameter DCD', MP, MQ shall cut off equal segments on this diameter measured from c, the centre. [Extension of one of Professor Catalan's.] 99

13423. (H. W. Curjel, M.A.)—If the opposite sides AB, DC of a quadrilateral meet in E, and AD, BC in F, and any straight lines EMN, PFQ are drawn through E and F, cutting AD, BC and AB, DC in M, N and P, Q, show that PN, MQ cut on AC, and PM, NQ on BD. 105

13424. (R. F. Davis, M.A.)—(1) If E be the centre of mean position for multiples l, m, n, at A, B, C, respectively, and ET the tangent to the circumcircle of ABC, prove that

$$l\mathrm{PA}^2 + m\mathrm{PB}^2 + n\mathrm{PC}^2 = (l+m+n)(\mathrm{PE}^2 - \mathrm{ET}^2).$$

Deduce that the equation $l\mathrm{PA}^2 + \ldots + \ldots = 0$ represents generally a circle cutting orthogonally the circumcircle, the trilinear coordinates of whose centre are $(l/a, m/b, n/c)$. When $l+m+n=0$ the above locus degenerates into a diameter of the circumcircle; as, for example,

$$(b^2-c^2)\,\mathrm{PA}^2 + \ldots + \ldots = 0$$

represents the join of the circum- and ortho-centres upon which lie, *inter alia*, the special points $\mathrm{PA}^2 : \mathrm{PB}^2 : \mathrm{PC}^2 = a^2 : b^2 : c^2$, or cot A : cot B : cot C.

(2) If ABC be a triangle, prove that the locus represented by

$$a^2(b^2-c^2)\,\mathrm{PA}^2 + b^2(c^2-a^2)\,\mathrm{PB}^2 + c^2(a^2-b^2)\,\mathrm{PC}^2 = 0$$

is the join of the circumcentre and the symmedian point ("diamètre de Brocard"); and upon it lie (*inter alia*) the point-pairs defined by

PA : PB : PC = cosec A : cosec B : cosec C (equilateral pedal triangles),

or = cot A : cot B : cot C

or $= (a^2 - b^2c^2/a^2)^{\frac{1}{2}} : \ldots :$ ... &c., &c. 113

13425. (J. J. Walker.)—Show that the vector equation to the surface generated by a transversal meeting the three lines through the terms of

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the vectors λ, μ, ν and parallel to the vectors α, β, γ respectively is

$$\xi S\alpha\beta\gamma = \Sigma\, \alpha \left\{(1+h_1)\, S\mu - h_1 S\nu\right\} V\beta\gamma, \quad \text{with} \quad 1+\Sigma h_1+\Sigma h_2 h_3 = 0.$$

..... 94

13427. (W. C. Stanham.)—If the pair of equations

$$lx+my+n \pm i\,(l'x+m'y+n') = 0 \quad [\text{where } i=(-1)^{\frac{1}{2}}]$$

are interpreted as representing the locus of the extremity of a line of length $\pm\left\{(l'x_1+m'y_1+n')/n\right\}$ units of length, drawn from a variable point (x_1, y_1) in the line $lx+my+n=0$, parallel to a fixed straight line L which neither lies in the plane of (x, y) nor is parallel to it, show that the equations of a pair of imaginary tangents to the circle $x^2+y^2=a^2$ from a point (x', y') within it represent two straight lines, and find their angle of inclination to the plane of (x, y) when L is perpendicular to this plane. Mention the limiting values of this angle for limiting values of a and positions of (x', y'). 92

13430. (Professor Sylvester.) — If the digits r in number of any integer N read from left to right be multiplied repeatedly by the first r terms of the recurring series $1, 4, 3, -1, -4, -3, \dot{1}, \dot{4}, \dot{3}, -\dot{1}, -\dot{4}, -\dot{3}$, show that, if the sum of these products be divisible by 13, so N will be, and not otherwise. 108

13446. (Editor.)—Prove that

$$4\left\{(a^2-b^2)cd+(c^2-d^2)ab\right\}^2+\left\{(a^2-b^2)(c^2-d^2)-4abcd\right\}^2=(a^2+b^2)^2(c^2+d^2)^2.$$

..... 109

13462. (Professor Crofton, F.R.S.)—Prove that 10^5+10 is divisible by 9^3+1..... 109

13464. (Professor A. Droz-Farny.)—Dans un triangle ABC on considère la circonférence Δ tangente en A à AB et qui passe par C et la circonférence Δ' tangente en A à AC et qui passe par B. Une transversale variable menée par A coupe Δ en α et Δ' en α'. Quel est le lieu du point milieu de $\alpha\alpha'$? 112

13466. (Professor M. J. M. Hill, M.A., D.Sc., F.R.S.)—Give proofs of (1) the 9th; (2) the 16th, with the 14th as corollary; (3) the 22nd; (4) the 23rd propositions of the Fifth Book of Euclid, based on Euclid's test for equal ratios, to replace his demonstrations, which are based on his test for unequal ratios; and hence show that all the propositions in the Sixth Book of Euclid can be proved without any reference direct or indirect to his test for unequal ratios. 102

13468. (Professor Neuberg.)—On considère les quadriques qui passent par une conique donnée Σ et par deux points donnés A, B. Trouver le lieu du centre et le lieu des extrémités du diamètre qui passe par le centre de Σ. 109

13478. (Morgan Brierley.)—ABCD is a given quadrilateral with the

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opposite sides AB, DC produced to meet in E; draw EQ to the intersection of the diagonals AC, BD. If a line LM be drawn through the middle of the sides AD, BC, it will pass through the middle N of EQ. 105

13484. (W. E. Jeffares, B.A.) — If ABC is a triangle, and O its circumcentre, through O draw any diameter LOM intersecting the circumcircle in L and M; find the locus of the intersection of the Simson-lines corresponding to L and M. 110

APPENDIX.

Unsolved Questions 121

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ERRATA IN VOL. LXVI.

Page 34, at foot, in Question 13077, the latter part of first result should be "$2\left\{\frac{3}{2^2}-\frac{5}{6^2}+\frac{11}{30^2}-\ldots\right\}$."

,, 42, at top, in Question 13133, the fifth result should be "$4a^2bcm_1/(b^4+c^4-a^4+2b^2c^2)$."

,, 100, second line, *for* "$2q^{2m-}$" *read* "$2q^{m-1}$."

,, 101, seventh line, *for* "$v=k-u$" *read* "$v=\mathrm{K}-u$."

,, ,, ninth line, *for* "$\cos\phi$" *read* "$\cos^2\phi$."

,, ,, eleventh line, *delete* Note.

,, ,, last line, *for* "$\psi\ -t)$" *read* "$\psi(-t)$."

,, 115, in figure, *for* "a" *read* "a_1."

,, ,, below figure, *for* "$ad\theta/dt=v'$" *read* "$a_1d\theta/dt=v'$."

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

13307. (Professor CROFTON, F.R.S.)—If m things are distributed among $n+x$ persons, in such a way that a particular set of n among them must each receive something, prove that the number of possible distributions is $N = \Delta^n x^m$.

Solution by Rev. W. A. WHITWORTH, M.A.

By *Choice and Chance*, Prop. XXI., the total number of distributions is $(x+n)^m$. Then, by Prop. XIV., the number of distributions in which n assigned persons shall receive something is

$$(x+n)^m - C_1^n (x+n-1)^m + C^n (x+n-2)^m - \&c. = \Delta^n (x^m).$$

[Mr. CURJEL gives this solution:—

Let $N = f(x, n)$. If A., one of the n persons, is removed from those who must receive something and placed among the others, the number of additional ways = the number of ways of distributing the m things so that A. receives nothing $= fx, (n-1)$;

$$\therefore \quad f(x, n) = f(x+1, n-1) - f(x, n-1) = \Delta f(x, n-1).$$

But $$f(x, 0) = x^m;$$

$$\therefore \quad f(x, 1) = \Delta x^m \quad \text{and} \quad N = f(x, n) = \Delta^n x^m.$$

The following solution is given by the PROPOSER:—

Put $N = u_{x,n}$; then, if a new person A. is introduced, *i.e.*, if x becomes $x+1$, the number of distributions is now $u_{x+1,n}$. Now A. either receives something or nothing; if the latter, the number is as before, $u_{x,n}$; if he gets something, we have $n+x+1$ persons out of whom a specified set of $n+1$ receive something, the number being $u_{x,n+1}$; hence

$$u_{x+1,n} = u_{x,n} + u_{x,n+1}; \quad \therefore \quad u_{x,n+1} = \Delta u_{x,n}.$$

Now, if $n = 0$, $$u_{x,0} = x^m;$$

$$\therefore \quad u_{x,1} = \Delta x^m, \quad u_{x,2} = \Delta u_{x,1} = \Delta^2 x^m,$$

and, finally, $$u_{x,n} = \Delta^n x^m.$$

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If m things are divided among n people so that each receives something,

$$N = \Delta^n 0^m ; \quad \text{if} \quad m = n, \quad N = 1 . 2 \ldots n.$$

If it be proposed, in dividing m things among n persons, to find the number of cases in which (at least) p unspecified persons each receive something, we have to subtract from the whole number n^m of possible cases, (1) those where one person receives all, (2) when two persons receive all, ... when $p-1$ persons receive all. It will thus appear that the answer is given by omitting the first $p-1$ terms of the series

$$n^m = (1+\Delta)^n 0^m = n + \frac{n(n-1)}{2}\Delta^2 0^m + \frac{n(n-1)(n-2)}{2 . 3}\Delta^3 0^m + \ldots .]$$

13328. (J. J. Walker, F.R.S.)—Show that the line of ultimate intersection of the planes $ax + by + cz = 1$, where the parameters a, b, c are connected by the relation $1 + \Sigma a + \Sigma bc = 0$, is a generator of the quadric $1 + \Sigma x + \Sigma yz = 0$.

Solution by W. C. Stanham, Professor Lampe, *and others.*

Since $\Sigma bc + \Sigma a + 1 = 0$, $(b+c+1)\,da + (c+a+1)\,db + \ldots = 0$ ... (1, 2).

Let (α, β, γ) be any point on the line of ultimate intersection (L), and denote the quadric by (Q); then we have

$$a\alpha + b\beta + c\gamma = 1, \quad \alpha da + \beta db + \gamma dc = 0 \ \ldots\ldots\ldots\ (3, 4).$$

The equation of (L) is given by

$$a(x-\alpha) + b(y-\beta) + c(z-\gamma) = 0, \quad da(x-\alpha) + db(y-\beta) + dc(z-\gamma) = 0,$$

and is therefore $(x-\alpha)/(bdc - cdb) = \&c. = \&c.,$

and from (2) and (4)

$$\{(a+b+1)\beta - (c+a+1)\gamma\}/da = \&c. = \&c.,$$

which gives, for the equation of (L),

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \ \ldots\ldots\ldots\ (5),$$

in which $l = \{(\Sigma a + 2)\alpha + (b+c+1)\}$, $m = \&c.$, $n = \&c.$

Let $\alpha' = lr + \alpha, \quad \beta' = mr + \beta, \quad \gamma' = nr + \gamma;$

then $(\alpha', \beta', \gamma')$ lies on (L).

And, if the coordinates of $(\alpha', \beta', \gamma')$ be substituted in the expression $\Sigma yz + \Sigma x + 1$, the result is

$$\{(\Sigma a + 2)r + 1\}^2 . \{\Sigma \beta\gamma + \Sigma \alpha + 1\}.$$

This will be zero if (α, β, γ) is one of the points at which (L) meets (Q), for all values of r. And, taking $r = -1/(\Sigma a + 2)$, we see that the point

$$x = -(b+c+1)/(\Sigma a + 2), \quad y = \&c., \quad z = \&c. .$$

is one of these points.

Since this is a real point, it follows that (L) is a generating line of (Q).

[The Rev. J. Cullen gives this solution:—The ultimate intersection

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of the planes lies on their envelope, and we have

$$x\,da + y\,db + z\,dc = 0, \quad \Sigma da + \Sigma\,(b+c)\,da = 0;$$

hence, by ordinary working, we find the envelope to be

$$1 + \Sigma x + \Sigma yz = 0.$$

Therefore the ultimate intersection is a generator of this quadric.]

13312. (Professor Genese.)—Let N be any integer, a, b, c, ..., the prime numbers less than $\surd$N, and

$$\phi\,(N) = \left\{1 - \Sigma\,(1/a) + \Sigma\,(1/ab) - \Sigma\,(1/abc) + \ldots)\,N\right\}$$

in every operation of which, such as N/a, N/b, N/ab, only the integral part of the quotient is taken. Then, in general, $\phi\,(N)$ + the number of the primes a, b, c ... = the number of primes (including unity) less than N; but, if N be the square of a prime number, the result of the count must be diminished by 1.

Solution by H. W. Curjel, M.A.; Professor Mukhopadhyay; *and others.*

Let N not be the square of a prime number. The number of numbers not greater than N which are divisible by $x = I\,(N/x)$. Now any of numbers 1, 2, ... N must be a prime $> \surd N$ or 1 or be divisible by at least one of a, b, c, ...; therefore number of primes $> \surd N$, $\not> N = \phi\,(N) - 1$; for, in subtracting $\Sigma I\,(N/a)$, we have taken away the number of numbers which are divisible by a, b, c, ..., one which is divisible by r different factors (a, b, c, ...) being counted r times; but it is counted $\frac{1}{2}\left\{r\,(r-1)\right\}$, $\left\{r\,(r-1)\,(r-2)\right\}/3!$, ... &c. times in $\Sigma I\,(N/ab)$, $\Sigma I\,(N/abc)$, ... &c.; therefore, $1-(1-1)^r$ times (taking account of signs), *i.e.*, only once. Hence the number of primes $\not> N = \phi\,(N)$ + number of primes a, b, c, ...

If $\surd N$ is a prime, the result must be diminished by unity, for N itself has been treated as a prime in the process; or the exception would be unnecessary if $\surd N$ were included among a, b, c, ...

13338. (Rev. T. T. Roach, M.A.)—A man riding from one point to another has to cross, first, a heavy sand on which he can only ride $6\frac{1}{2}$ miles per hour, and then firm grass on which he can ride 10 miles per hour. Find the shortest time in which he can ride, given that the perpendicular distances of the two points from a straight road separating the two districts are 4 and 5 miles, respectively, and that the feet of the perpendiculars are 15 miles apart.

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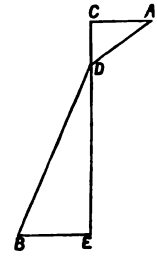
Solution by Rev. J. CULLEN, B.A.; Professor SWAMINATHA AIYAR; *and others.*

Let x be the distance between the point where he crosses the road and the foot of the shorter perpendicular, and t the time. Then



$$t = (x^2+16)^{\frac{1}{2}}/6\tfrac{1}{2} + \{25+(15-x)^2\}^{\frac{1}{2}}/10.$$

$dt/dx = 0$ gives a quartic which may be written

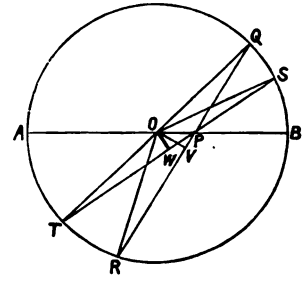
$$\{77x^2(x-27)+13520(x+5)\}(x-3) = 0.$$

The first factor does not vanish for any value of x between 0 and 15. Hence we have $x = 3$ and the respective distances 5 and 13; whence $t = 2{\cdot}0692$ hrs., *i.e.*, 2 hrs. 4 mins. 9·12 secs.

13314. (Professor MATZ.)—Find the average area of the random sector whose vertex is a random point in a given circle.

Solution by D. BIDDLE, Professor NATH COONDOO, *and others.*

Assuming the random sector in question to be constituted by two random lines and the intercepted portion of the given circumference, the lines being drawn in any direction through a random point P, and separated by any angle not exceeding π, it would seem that a choice of four values is given on every occasion. Thus we have QPS, RPT, QPT, RPS; and the mean area of these sectors is $\frac{1}{4}\pi r^2$ for every position of P, and for every direction of the lines. Moreover, all the values that would be summed by the method of integration are thus covered. The answer therefore is $\frac{1}{4}\pi r^2$.



13303. (Professor NEUBERG.)—Sur les côtés BC, CA, AB d'un triangle ABC, on marque les sommets A′, B′, C′ d'un triangle A′B′C′ dont les côtés a', b', c' vérifient la relation donnée $aa'^2+\beta b'^2+\gamma c'^2 = 0$. ($\alpha$) Si l'on donne le point C′, la droite A′B′ enveloppe une courbe de la quatrième classe. (β) Si l'on donne les points A′, B′, le point C′ occupe sur AB deux positions C'_1, C'_2. Les points C'_1, C'_2 coïncident, lorsque la droite A′B′ enveloppe une certaine courbe de la quatrième classe. (γ) Enfin, si les points A′, B′, C′ sont sur une même droite, cette droite enveloppe une certaine courbe de la quatrième classe.

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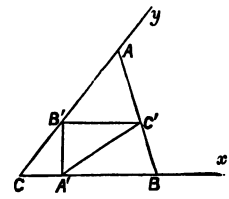
Solution by H. W. Curjel, M.A.; Prof. Nath Coondoo; *and others.*

Take CB, CA as axes of x and y; let

$CA' = 1/l$, $CB' = 1/m$, $AC' = k$.



Then equation to A'B' is $lx + my = 1$, and the condition $aa'^2 + \beta b'^2 + \gamma c'^2 = 0$ becomes

$$\begin{aligned}
&a\left\{k^2 + \left[b - (1/m)\right]^2 \right.\\
&\qquad \left. - 2k\left[b - (1/m)\right]\cos A\right\}\\
&+\beta\left\{(c-k)^2 + \left[a - (1/l)\right]^2 \right.\\
&\qquad \left. - 2(c-k)\left[a - 1/l\right]\cos B\right\}\\
&+\gamma\left\{(1/l^2) + (1/m^2) - \left[(2\cos C)/(lm)\right]\right\} = 0 \quad \ldots\ldots\ldots\

Solution by Rev. J. CULLEN, B.A.; Professor SARKAR; *and others.*

$APB = 40°$ and $A'PB' = 50°$;

$\therefore \quad x/a = \sin(\phi-40)/\sin 40$

$= \sin(50-\psi)/\sin 130.$

Also $BAB'/AB'A = (\phi+180-\psi)/180$

$= 10/9;$

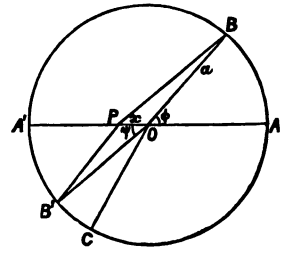
$\therefore \quad \phi-\psi = 20°.$

Hence $x/a = \sin(\psi-20)/\sin 40$

$= \sin(50-\psi)/\cos 40.$

$\therefore \quad \psi = 33° 39' 25''$

and $\quad x = \cdot 3275998a.$



If α be the angle described by B round the centre O in a given time, and β the angle between A and B, then we have

$$(\alpha+\beta-\phi)/\alpha = 9/10; \quad \therefore \quad \alpha = 10\phi-10\beta.$$

When they meet, $\beta = 0$. Hence

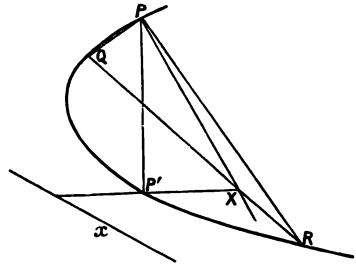
$\alpha = 536° 34' 10''$ and $\angle AOC = \alpha-360-\phi = 122° 54' 45''$;

C being the required point.

13327. (R. KNOWLES, B.A.)—PQ, PR are chords of a parabola at right angles; prove that (1) the locus of the pole of QR is a right line; (2) as P varies, the envelope of this locus is an equal parabola.

Solution by H. W. CURJEL, M.A.; Professor A. DROZ-FARNY; *and others.*

If PP′ is a chord perpendicular to the axis, QR always passes through X, the intersection of the normal at P with the diameter through P′. Therefore locus of the pole of QR is x, the polar of X; also x is clearly parallel to the tangent at P′, and its distance from P′, measured along the diameter through P′ = P′X $= 4a$. Therefore the envelope of x is an equal parabola with the same axis, and distance between the vertices $= 4a$.



13290. (Rev. J. CULLEN, B.A.)—One end of a heavy beam lying in a horizontal plane is fastened by a hinge to the bottom of an upright post. To the other end of the beam is attached a light rope passing over a pulley at the top of the post. Find the curve in which a weight

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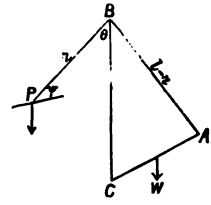
fastened to the rope must slide in order that the system may be in equilibrium for all positions of the beam; neglecting friction and assuming the weight and beam to move in the same vertical plane.

Solution by H. W. Curjel, M.A.; Professor Gopalachanar; *and others.*

Let BC be the post, CA the beam (weight W), P the weight, and l the length of the rope. Let BP $= r$, $\angle$ PBC $= \theta$, and angle between the tangent at P to the curve and BP $= \psi$.



$$\text{Tension of rope} = \frac{W \tfrac{1}{2} b \sin C}{b \sin A} = \frac{W.c}{2a} = \frac{W(l-r)}{2a},$$

and $$\cos(\theta+\psi) = \frac{W}{2Pa}(l-r)\cos\psi = \frac{2(l-r)}{h}\cos\psi \text{ (say).}$$

Therefore locus of P is given by

$$\cos\theta\frac{dr}{ds} - r\sin\theta\frac{d\theta}{ds} = \frac{2(l-r)}{h}\frac{dr}{ds},$$

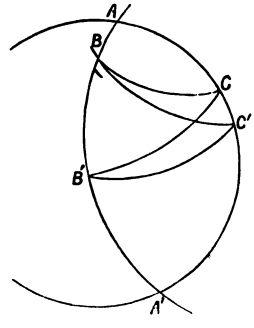
i.e., $h(r\cos\theta + k) = 2lr - r^2$, or $(x^2+y^2+hx+hk)^2 = 4l^2(x^2+y^2)$.

13219. (R. W. D. Christie.)—Prove that the sum of the angles in a right-angled primitive spherical triangle is less than three right angles.

Solution by Professors Jeffares, Nath Coondoo, *and others.*

Let ABC be a spherical right-angled triangle, having A the right angle. With A as centre and an arc of 90° as radius, describe on the sphere the arc B′C′, meeting the sides of the triangle ABC in B′, C′. Join B′C and BC′ (of course with arcs of great circles). Since B′A = 90° and B′C′ = 90° (since it measures the angle A), therefore B′ is the pole of AC′ and C′ the pole of AB′. Now B′C is a secondary circle to AC′, and therefore at right angles to it. $\angle$ ACB is less than a right angle; similarly ABC is less than a right angle;

$\therefore$ $\angle A + \angle B + \angle C > 3$ right angles.



I think the proposition is only true for right-angled triangles whose sides are each less than a quadrant. For, if we take the colunar triangle A′BC, we see the two angles A′BC

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and A'CB are together greater than two right angles, and $\angle A'$ is a right angle. $\therefore$ A'BC + A'CB + $\angle A' > 3$ right angles.

$\left[\frac{\sin A}{\sin C} = \frac{\sin a}{\sin c}$ and $\sin C = 1$; $\therefore \sin A = \frac{\sin a}{\sin c} \not> 1$;

$\therefore \sin a \not> \sin c$; $\therefore a \not> c$; $\therefore A \not> C \not> 90°$; similarly for B.$]$

13311. (Professor SCHWATT.) — The SIMSON-line belonging to one point of intersection of BROCARD's diameter of a triangle with the circumcircle of this triangle is either parallel or perpendicular to the bisector of the angle formed by the side BC of the triangle ABC and the corresponding side B'C' of BROCARD's triangle.

Solution by H. W. CURJEL, M.A.; Professor GOPALACHANAR; *and others.*

Let POP' be any diameter of the circumcircle of $\triangle ABC$. Let ZXY be the SIMSON-line of P, and let ZP, PY meet the circle in C_1, B_1. Then

$\angle PZX = \angle PBX = \angle PAC$,

and $\angle ZPX = \angle ABC = \angle APO$,

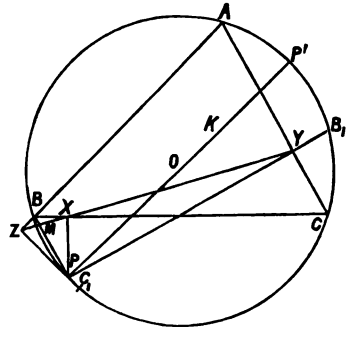
and PB is a diameter of $\odot BZPX$;

therefore figures BZPX, P'APO are similar; therefore, if PB, PP cut XZ, CA in M, N,

PM : MB = PN : NP'

= PY : YB_1;

therefore BB_1, ZY are parallel. But CC_1 is parallel to BB_1, for



$\angle C_1PB_1 = \angle BAC$, and hence arc BC_1 = arc CB_1.

Therefore bisector of angle between BC and B_1C_1 is parallel to ZXY, the SIMSON-line of P, and therefore perpendicular to the SIMSON-line of P'.

But, if the symmedian point K lies in POP' between O and P', B_1C_1 is clearly parallel to B'C'. Hence the proposition.

13324. (W. J. DOBBS, M.A.)—Four fine light rods are smoothly jointed at their extremities to form a quadrilateral which can be inscribed in a circle. The opposite joints are connected by two fine light strings in a state of tension. Prove that the thrusts in the rods and the tensions of the strings are proportional to the opposite sides and diagonals of the quadrilateral, respectively.

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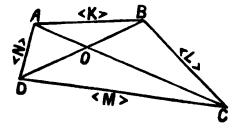
Solution by Rev. J. CULLEN, B.A.; Professor NATH COONDOO; *and others.*

Let K, &c., be the thrusts along the rods P and Q the tensions along AC and BD.



We have $K = P\frac{OD \cdot AB}{OA \cdot BD}$

(WALTON'S *Problems*, p. 101);

also $M = P\frac{OB \cdot CD}{OC \cdot BD}$.

$\therefore$ $K : M = OD \cdot AB \cdot OC : OA \cdot CD \cdot OB = OD : OB = CD : AB$,

since the triangles AOB and COD are similar, the points A, B, C, and D being concyclic. In like manner, $L : N = AD : BC$.

Also $P : Q = BD \cdot OA \cdot OC : AC \cdot OB \cdot OD = BD : AC$.

13152. (Rev. T. WIGGINS, B.A.)—Given a conic with focus S, and P, Q any two points on it; if the sum of SP, SQ is constant, find the locus of the middle point of PQ.

Solution by Professors RADHAKRISHNAN, A. DROZ-FARNY, *and others.*

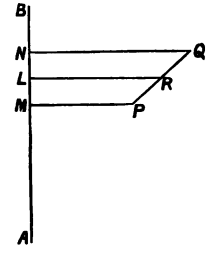
If AB be the directrix,

$$SP = e \cdot PM,$$
$$SQ = e \cdot QN.$$

If R be the middle point of PQ,

$$RL = \tfrac{1}{2}(PM + QN)$$
$$= \tfrac{1}{2}\left(\frac{SP}{e} + \frac{SQ}{e}\right)$$
$$= \frac{1}{2e} \cdot \text{a constant.}$$

Therefore the locus of R is a line parallel to AB.



13206 & 13267. (R. CHARTRES.)—If a number a be divided at random into n parts, find the mean value of the mth power of one of the parts, with its geometrical application.

(13267.) (R. CHARTRES.)—If a point P be taken at random in a triangle, and lines PQ, PR be drawn parallel to the sides, prove that the mean value of $QR = \frac{1}{3}a$, and the mean value of $QR^2 = \frac{1}{6}a^2$, or the mean value of $PQR = \frac{1}{9}\Delta$; also, if three points P, Q, R be taken at random in a triangle ABC, and DE, DF, EF be drawn through them parallel to

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the sides, then the mean value of PQR = ¼DEF, of DEF = ⅓Δ; therefore of PQR = $\frac{1}{12}\Delta$.

Solution by R. CHARTRES, Professor GOPALACHANAR, *and others.*

Let a be divided at random into n parts, x, y, z, &c.; then

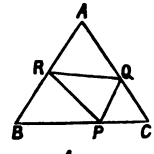
$$a^m = (x+y+z+\ldots)^m, \quad \text{which has} \quad \frac{n(n+1)\ldots(n+m-1)}{m!} \text{ terms;}$$

therefore the mean value of any one term, as x^m,

$$= a^m + \frac{n(n+1)\ldots(n+m-1)}{m!}.$$

Cor. 1. If $n=2$, $m=2$, then the mean value of $x^2 = \frac{1}{6}a^2$ and of $xy = \frac{1}{6}a^2$.

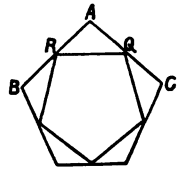
Cor. 2. If $n=3$, $m=2$, the mean value of $x^2 = \frac{1}{6}a^2$ and of $xy = \frac{1}{12}a^2$.



Geometrical Application.—If three points P, Q, R be taken at random in the sides of a triangle, one in each side, then the mean value of each triangle, as AQR, $= \frac{1}{4}\Delta$.

Hence, if n points be taken at random, one in each side of a regular polygon of n sides, the mean value of the inscribed figure

$$= \cos^2(\pi/n),$$



if the area of the original polygon be unity, since the mean value of AQR

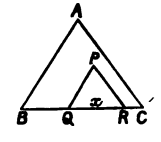
$$= \frac{1}{4}\text{ABC}.$$

If a point P be taken at random within a triangle, and PQ, PR be drawn parallel to the sides, then, since the base consists of three parts, the mean value of x $= \frac{1}{3}a$,

and therefore of x^2 $= \frac{1}{6}a^2$; (Cor. 2)

$\therefore$ mean value of PQR $= \frac{1}{6}\Delta$,

similar triangles.



If three points P, Q, R be taken at random within a triangle, then, drawing through them lines parallel to the sides, the mean value of PQR

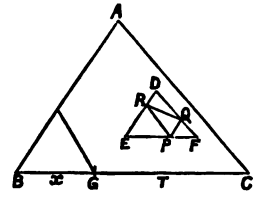
$$= \frac{1}{4}\text{DEF},$$

and the mean value of DEF

$$= \frac{1}{4}\Delta, \quad \text{(Cor. 1)}$$

making BG = EF = x;

$\therefore$ mean value of PQR = $\frac{1}{12}\Delta$.



[For the application of "mean value" to integration, see Quest. 13171.]

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13197. (Professor FINKEL.) — Two points are taken at random on the circumference of a semicircle. Find the chance that their ordinates fall on either side of a point taken at random on the diameter.

Solution by H. W. CURJEL, M.A.; Professor NATH COONDOO; *and others.*

Since $$\int_0^1 \cos^{-1} x\, dx = -\int_0^{\frac{1}{2}\pi} \sin\theta\, d\theta,$$

we see that the chance that the ordinates are on opposite sides of the random point

$$= \frac{2\int_0^{\frac{1}{2}\pi}\int_\theta^\pi\int_0^\phi \sin\theta\, d\phi\, d\psi\, d\theta}{\int_0^{\frac{1}{2}\pi}\int_0^\pi\int_0^\pi \sin\theta\, d\phi\, d\psi\, d\theta} = \frac{2}{\pi^2}\int_0^{\frac{1}{2}\pi} \theta(\pi-\theta)\sin\theta\, d\theta = \frac{4}{\pi^2}.$$

4824. (Professor WOLSTENHOLME, F.R.S.)—A large area is to be paved with circular discs of black marble which are to touch; the interstices to be filled up with slabs of white marble. Each piece of pavement must be cut from a square slab: if the price of the slabs per foot of area vary as the length of a side, and if the cost of cutting the marble be a shillings per linear foot, and the price of a slab one foot square be b shillings, show that, in order that the area may be paved with least expense, the radius of each disc must be $(2\pi a^{-1})^{\frac{1}{2}}$.

Solution by D. BIDDLE.

Assuming, since nothing appears to the contrary, black and white marble to be equally valuable, we have as the price per disc when this is one foot in diameter, the cost of the interstitial white marble being added,

$$(1 + \sqrt{\tfrac{1}{3}})\, b + 2\pi a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

The side of the square from which the white (quadruply concaved) interstice is cut is $\sqrt{\frac{1}{3}}$, but, as the value per square foot becomes thereby $\sqrt{\frac{1}{3}}b$, the cost of the material is $\frac{1}{3}\sqrt{\frac{1}{3}}b$, whilst its cost for cutting equals that of the black marble, each being πa.

Let n^2 = the number of black discs; then the total cost of the slabs is as $1/n$, and the total cost of cutting as n; or, to put it otherwise in reference to the radius, the total cost of the slabs is as r, and the total cost of cutting as $1/r$. Consequently, $(1 + \sqrt{\frac{1}{3}})\, br + (2\pi a/r)$ = a minimum ... (2).

Taking, in reference to r, the differential coefficients of (2), we obtain

$$(1 + \sqrt{\tfrac{1}{3}})\, b - (2\pi a/r^2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (3);$$

whence $r = \left\{\dfrac{2\pi a}{(1+\sqrt{\frac{1}{3}})\, b}\right\}^{\frac{1}{2}}$ of the original radius (which was 6 inches),

$= \left\{\dfrac{\pi a}{(2+\sqrt{\frac{1}{3}})\, b}\right\}^{\frac{1}{2}}$ of a foot, a result differing from that in the Question.

[Mr. CURJEL gives this solution :—"Let the radius of the circular discs

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be x feet. Then the sides of the black and of the white slabs must be $2x$ and $x\sqrt{2}$ feet, respectively;

$\therefore$ cost per $(2x)^2$ sq. ft. $= 4x^2 \times 2xb + 2x^2 \times x\sqrt{2} \times b + 4\pi xa$;

$\therefore$ cost per sq. ft. $= 2bx + (bx/\sqrt{2}) + (\pi a/x) = \mu$;

therefore, for critical value,

$$\left(2 + \frac{1}{\sqrt{2}}\right) b - \frac{\pi a}{x^2} = \frac{du}{dx} = 0, \quad \text{or} \quad x = \left\{\frac{\sqrt{2}\pi a}{(\sqrt{8}+1)\,b}\right\}^{\frac{1}{3}} \text{feet},$$

which gives a minimum, for d^2u/dx^2 is positive."]

[Mr. SIMMONS, being asked to comment, states:—"Of the two solutions above, the results are identical and apparently correct. The result given in the Question is evidently wrong, as the correct result ought clearly to be a function of b as well as a, and ought to be large or small according as a is large or small compared with b."]

13333. (J. J. BARNIVILLE, B.A.)—Prove that

$$\frac{1}{1.2.3} + \frac{1}{1.3.5} + \frac{1}{2.5.8} + \frac{1}{3.8.13} + \ldots = \tfrac{1}{4},$$

$$\frac{1}{1.2.3} - \frac{1}{1.3.5} + \frac{1}{2.5.8} - \ldots = -\tfrac{3}{4} + \tfrac{1}{3} + \tfrac{1}{8} + \tfrac{1}{8} + \ldots.$$

Solution by H. W. CURJEL, M.A.; Professor NATH COONDOO; *and others.*

If $u_0, u_1, u_2, u_3, u_4, \ldots = 1, 1, 2, 3, 5, 8, 13, 21, \ldots,$

$u_n = u_{n-1} + u_{n-2}$ and $u_{n+2} = 2u_n + u_{n-1}$; hence the n^{th} term of first series

$$= \frac{u_n}{u_{n-1}\,u_n\,u_{n+1}\,u_{n+2}} = \tfrac{1}{2}\frac{1}{u_{n-1}\,u_n\,u_{n+1}} - \tfrac{1}{2}\frac{1}{u_n\,u_{n+1}\,u_{n+2}} = \tfrac{1}{2}w_n - \tfrac{1}{2}w_{n+1};$$

$\therefore$ sum of first series $= \tfrac{1}{2}w_1 = \tfrac{1}{4}$.

Again,
$$u_{n-1}\,u_n\,u_{n+1}\left(\frac{1}{u_{n+1}} + \frac{1}{u_n} - \frac{1}{u_{n-1}}\right)$$
$$= u_{n-1}\,u_n - u_{n+1}\,(u_n - u_{n-1}) = u_n\,u_{n-1} - (u_n + u_{n-1})\,u_{n-2}$$
$$= -u_{n-2}\,u_{n-1}\,u_n\left(\frac{1}{u_n} + \frac{1}{u_{n-1}} - \frac{1}{u_{n-2}}\right) = (-1)^{n+1}\mathrm{A}.$$

Putting $n = 3$, $\mathrm{A} = 2.3.5\,(\frac{1}{5} + \frac{1}{3} - \frac{1}{2}) = 1$; hence the second series

$$= \tfrac{1}{2}w_1 - w_2 + w_3 - w_4 + \ldots$$
$$= \tfrac{1}{4} + (-\tfrac{1}{1} + \tfrac{1}{3} + \tfrac{1}{8}) + (-\tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{5}) + (-\tfrac{1}{5} + \tfrac{1}{8} + \tfrac{1}{8}) + \ldots$$
$$= -\tfrac{3}{4} + \tfrac{1}{3} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{13} + \ldots.$$

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13258. (Professor SIVERLY.)—Of all cones which develop an ellipsoid and have their bases in the tangent plane at a given point P, and are of the same altitude, that is the least which has its vertex in the diameter through P; and of all which have their vertices in this diameter, that is the least whose axis is twice this diameter.

Solution by Rev. J. CULLEN, B.A.; Professor MATZ; *and others.*

The equation of the ellipsoid referred to conjugate diameters is

$$\Sigma x^2/a^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1);$$

taking P (0, 0, c) for the origin, (1) becomes

$$\Sigma x^2/a^2 - 2z/c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

The enveloping cone, having (x_1, y_1, z_1) for its vertex, is therefore

$$(\Sigma x^2/a^2 - 2z/c)(\Sigma x_1^2/a^2 - 2z_1/c) - \{\Sigma x x_1/a^2 - (z+z_1)/c\}^2 = 0 \quad \ldots \quad (3);$$

putting $z = 0$ and $z_1 = \gamma$, we obtain

$$x^2/a^2(y_1^2/b^2 + \gamma^2/c^2 - 2\gamma/c) + y^2/b^2(x_1^2/a^2 + \gamma^2/c^2 - 2\gamma/c) - 2xy\, x_1y_1/a^2b^2 + 2xx_1\gamma/a^2c + 2yy_1\gamma/b^2c - \gamma^2/c = 0,$$

the ellipse formed by the intersection of the tangent plane at P with (3). Its area, and therefore the volume of the cone, is easily seen to be a minimum when $x_1 = 0$ and $y_1 = 0$, *i.e.*, when the vertex is in the diameter through P.

The area of $(x^2/a^2 + y^2/b^2)(\gamma^2/c^2 - 2\gamma/c) - \gamma^2/c^2 = 0$

varies as $\gamma/(\gamma - 2c)$. Hence, if V be the volume of the cone,

$$V = k\gamma^2/(\gamma - 2c).$$

Therefore $dV/d\gamma = 0$ gives $\gamma = 4c$ = 2 diameter through P.

[Another solution of this Question is given in Vol. LXVI., p. 113.]

13326. (R. F. DAVIS, M.A.)—If A, B, C be the angles of a triangle, prove they satisfy a relation of the form $\cos^2\theta + \lambda\cot\theta = P$.

Solution by Professors DROZ-FARNY, FINKEL, *and others.*

Comme $$\cos^2\theta = \frac{\cot^2\theta}{1+\cot^2\theta},$$

la relation $\cos^2\theta + \lambda\cot\theta = P$ deviendra

$$\frac{\cot^2\theta}{1+\cot^2\theta} + \lambda\cot\theta = P \quad \text{ou} \quad \cot^3\theta + \frac{1-P}{\lambda}\cot^2\theta + \cot\theta - \frac{P}{\lambda} = 0.$$

Les trois racines de cette équation $\cot\theta'$, $\cot\theta''$, $\cot\theta'''$ vérifiant la relation

$$\cot\theta'\cot\theta'' + \cot\theta'\cot\theta''' + \cot\theta''\cot\theta''' = 1,$$

on a bien $$\theta' + \theta'' + \theta''' = 180°;$$

d'où le théorème.

Remarque : L'angle ϕ de BROCARD du triangle est alors donné par la relation $$\cot\phi = \frac{P-1}{\lambda}.$$

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11895. (Professor MUKHOPADHYAY, M.A.) — PLINY informs us that "THALES determined the cosmical setting of the Pleiades to have happened in his time 25 days after the Autumnal Equinox." Supposing the longitude of the brightest star in that constellation to have been, at the beginning of the year 1760, according to Dr. BRADLEY, ♉ 26° 38′ 34″, and its latitude 4° 1′ 36″, the precession of the Equinox 50·34″, annually, and the latitude of Miletus 37° 30′, determine the time when THALES lived.

Solution by C. BICKERDIKE, Professor KRISHMACHANDRA DE, *and others.*

As the Pleiades set with the Sun 25 days after the Autumnal Equinox, we may without appreciable error assume that they were at the Equinox together on the 21st, and that, therefore, their longitudes would be the same.

Now, if we take this for our origin, we have

the longitude of the constellation ♉ 26° 38′ 34″, and 30° 0′ 0″ + 26° 38′ 34″ = 56° 38′ 34″ for the year 1760.

Hence $90° 0′ 0″ - 56° 38′ 34″ = 33° 21′ 26″ = 120086″$,

which, divided by 50·34″, gives 2383 years, and 2383 − 1760 = 623 years B.C.

VINCE states that THALES lived about 600 years A.C., and died 549 A.C.; BLACK states in his *Encyclopædia* that he died 545 B.C., at the age of 90.

13167. (Professor DE LONGCHAMPS.)—Par le centre d'une ellipse E, on fait passer une circonférence Δ. Soient A, B, C trois des points communs aux courbes E, Δ; on suppose que les normales en ces points concourent en P. Trouver (1) le lieu décrit par le centre de Δ; (2) le lieu décrit par le point P. Ces lieux sont des ellipses.

Solution by Professors DROZ-FARNY, RADHAKRISHNAN, *and others.*

Soient (α, β) les coordonnées de P. De P on mène les quatre normales possibles à l'ellipse; soient A, B, C, et D leurs pieds. On sait que les trois points A, B, C et le point D′ diamétralement opposé de D sont sur une circonférence, la circonférence de JOACHIMSTHAL. Soient x', y' les coordonnées de D. En posant

$$u = a^2 + (b^2\beta/y') = b^2 + (a^2\alpha/x')$$

(d'après M. DE LONGCHAMPS, *Géométrique Analytique*, p. 402), l'équation de cette circonférence peut s'écrire

$$x^2 + y^2 + xx' + yy' = u\left\{(xx'/a^2) + (yy'/b^2) + 1\right\}.$$

Si elle doit passer par l'origine $u = 0$, de là

$$y' = -(b^2\beta/a^2), \quad x' = -(a^2\alpha/b^2),$$

et comme ces coordonnées vérifient l'équation

$$(x'^2/a^2) + (y'^2/b^2) = 1,$$

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on a, pour le lieu de P, l'ellipse

$$(a^2\alpha^2/b^4) + (b^2\beta^2/a^4) = 1.$$

Le cercle de JOACHIMSTHAL ayant, dans ce cas, pour équation

$$x^2 + y^2 + xx' + yy' = 0,$$

les coordonnées de son centre sont $x = -\frac{1}{2}x'$, $y = -\frac{1}{2}y'$; d'où, pour le lieu de ce point, l'ellipse $x^2/a^2 + y^2/b^2 = \frac{1}{4}$.

13213. (J. J. BARNIVILLE, B.A.)—Prove that

$$\sqrt{5} = \left(\frac{2}{1.1}\right)^2 - \left(\frac{3}{1.2}\right)^2 + \left(\frac{5}{2.3}\right)^2 - \left(\frac{8}{3.5}\right)^2 \quad \ldots\ldots\ldots (S_1),$$

$$= 2 + \left(\frac{1}{1.2}\right)^2 - \left(\frac{1}{2.3}\right)^2 + \left(\frac{2}{3.5}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots (S_2).$$

Solution by Professor SANJÁNA; H. W. CURJEL, M.A.; *and others.*

By the well-known formula

$$\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} + \ldots = \frac{1}{u_1 -} \frac{u_1^2}{u_1 + u_2 -} \frac{u_2^2}{u_2 + u_3 -} \ldots,$$

we get $\frac{1}{1.1} - \frac{1}{1.2} + \frac{1}{2.3} - \frac{1}{3.5} + \frac{1}{5.8} - \ldots = \frac{1}{1-} \frac{1}{-1-} \frac{4}{4-} \frac{36}{-9-} \ldots$

$$= \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \ldots = \frac{\sqrt{5}-1}{2}.$$

Now $S_1 = \frac{1}{1^2} + \frac{1}{1^2} + \frac{2}{1.1} - \frac{1}{1^2} - \frac{1}{2^2} - \frac{2}{1.2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{2}{2.3}$

$$- \frac{1}{3^2} - \frac{1}{5^2} - \frac{2}{3.5} + \ldots = \frac{1}{1^2} + 2\left\{\frac{1}{1.1} - \frac{1}{1.2} + \frac{1}{2.3} - \frac{1}{3.5} \ldots\right\}$$

$$= 1 + \sqrt{5} - 1 = \sqrt{5}.$$

And $S_2 = 2 + \frac{1}{1^2} + \frac{1}{2^2} - \frac{2}{1.2} - \frac{1}{2^2} - \frac{1}{3^2} + \frac{2}{2.3} + \frac{1}{3^2} + \frac{1}{5^2} - \frac{2}{3.5} \ldots$

$$= 2 + \frac{1}{1^2} - 2\left\{\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.5} \ldots\right\}$$

$$= 3 - 2\left(1 - \frac{\sqrt{5}-1}{2}\right) = 3 - 2 + \sqrt{5} - 1 = \sqrt{5}.$$

13225. (Professor SANJÁNA, M.A.)—From points on a fixed straight line tangents are drawn to a given conic, and triangles are formed by them with their chords of contact. Prove that the locus of the median

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points (centroids) of the triangles is a parabola when the conic is parabolic, and a cubic curve when the conic is central. Also, trace the curve $3bx - b^2 = 2a^2x^2/(x^2+y^2)$, which is the locus when the conic is a circle. [This cubic is the inverse of an ellipse, and is frequently met with.]

Solution by Rev. J. CULLEN, B.A.; Professor FINKEL; *and others.*

Taking the straight line for the axis of y the conic to be

$$(a, b, c, f, g, h)(X, Y, 1)^2 = 0 \quad \ldots\ldots\ldots\ldots (1),$$

and (x, y) the centroid, the triangle is then formed by joining $(0, \beta)$, (x_1, y_1), (x_2, y_2), where x_1, x_2 and y_1, y_2 are the roots of the quadratics formed by eliminating Y and X between (1) and

$$X(h\beta+g) + Y(b\beta+f) + f\beta + c = 0,$$

the polar of $(0, \beta)$; hence

$$x_1 + x_2 = 2G\gamma/(C\gamma - \Delta), \quad y_1 + y_2 = (4F\gamma - 2\beta\Delta)/(C\gamma - \Delta),$$

where C, F, and G are the minors of c, f, and g in Δ, the discriminant of (1) and $\gamma = b\beta^2 + 2f\beta + c$.

But $3x = x_1 + x_2$ and $3y = y_1 + y_2 + \beta$. Therefore, eliminating β and γ, we obtain the required locus

$$(3Cx - 2G)\{b(Gy - 2Fx)^2 + 2f(Gy - 2Fx)(G - Cx) + c(G - Cx)^2\} - 3x\Delta(G - Cx)^2 = 0 \quad \ldots\ldots\ldots\ldots (2),$$

which reduces to a parabola if C or $ab - h^2 = 0$, *i.e.*, if (1) is a parabola.

For the circle $X^2 + Y^2 - 2bX + b^2 - a^2 = 0$, (2) becomes

$$(x^2+y^2)(3bx - b^2) = 2a^2x^2$$

in transforming the origin to the centre. This cubic is symmetrical with regard to the axis of x. The origin is an acnode. There is one real asymptote $x = b/3$, two points of inflexion determined by

$$x = 2b(2a^2 + b^2)/3(a^2 + 2b^2),$$

and the whole curve lies between $x = b/3$ and $x = (2a^2 + b^2)/3b$.

13190. (R. W. D. CHRISTIE.) — Let 1/P, where P is any integer ending in the digits 1, 3, 7, or 9, have $r_1, r_2, \ldots r_p$, remainders; then

$$r^p - 1 = P.Q, \quad e.g., \quad 24^7 - 1 = 239\,(M).$$

Solution by H. W. CURJEL, M.A.; Professor SWAMINATHA AIYAR; *and others.*

Let R be the radix of the scale of notation, and P prime to R. If there are p remainders, $R^p \equiv 1 \bmod P$, also $r_s \equiv R^s \bmod P$;

$$\therefore \quad r_s^p \equiv R^{ps} \equiv 1 \bmod P.$$

If R = 10, the last digit of P will clearly be 1, 3, 7, or 9.

[A conjugate theorem is $\rho^p + 1 \equiv P.Q$ if $r + \rho \equiv P$.]

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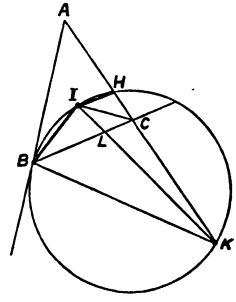
13173. (W. J. Dobbs, M.A.)—A circle drawn touching AB at B and and passing through I the in-centre of the triangle ABC, meets AC in H and K. Prove that IC bisects the angle HIK.

Solution by R. F. Davis, M.A.; E. W. Todd; *and others.*



Let IK intersect BC in L. Then

$$\angle \mathrm{IBL} = \mathrm{IBA} = \mathrm{IKB};$$

therefore the triangles IBL, IKB are similar, and

$$\angle \mathrm{ILB} = \mathrm{IBK} = \mathrm{IHA}.$$

Taking $\frac{1}{2}$C from each angle,

$$\angle \mathrm{CIL} = \mathrm{CIH}.$$

13302. (R. W. D. Christie.)—Prove that the tangents and cotangents of the half angles of any triangle are the roots of the equations

$$sx^3-(4\mathrm{R}+r)x^2+sx-r=0, \quad rx^3-sx^2+(4\mathrm{R}+r)x-s=0,$$

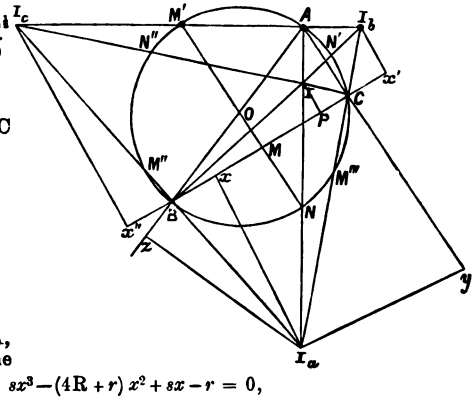
where the letters have the usual meanings.

Solution by W. E. Jeffares, B.A.; H. W. Curjel, M.A.; *and others.*

$$\tan\tfrac{1}{2}\mathrm{A}\tan\tfrac{1}{2}\mathrm{B}\tan\tfrac{1}{2}\mathrm{C} = \left\{\frac{(s-a)(s-b)(s-c)}{s^3}\right\}^{\frac{1}{2}} = \frac{\mathrm{S}}{s^2} = \frac{r}{s};$$

$$\tan\tfrac{1}{2}\mathrm{A}+\tan\tfrac{1}{2}\mathrm{B}+\tan\tfrac{1}{2}\mathrm{C} = \frac{bc+ca+ab-s^2}{\mathrm{S}} = \frac{r+4\mathrm{R}}{s};$$

$$\tan\tfrac{1}{2}\mathrm{B}\tan\tfrac{1}{2}\mathrm{C}+\tan\tfrac{1}{2}\mathrm{C}\tan\tfrac{1}{2}\mathrm{A}+\tan\tfrac{1}{2}\mathrm{A}\tan\tfrac{1}{2}\mathrm{B} =$$



Therefore tan A, tan $\frac{1}{2}$B, tan $\frac{1}{2}$C are he roots of

$$sx^3-(4\mathrm{R}+r)x^2+sx-r=0,$$

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and cot ½A, cot ½B, cot ½C of

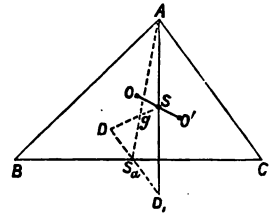
$$rx^3 - sx^2 + (4R + r)x - s = 0.$$

Corollaries.—(1) Multiply the roots of the first equation by s, and we get $x^3 - (4R + r)x^2 + s^2x - rs^2 \equiv 0$, whose roots are r_1, r_2, r_3.

( ) Multiply the roots of the second equation by r, and we get $-sx^2 + r(4R + r)x - r^2s \equiv 0$, with roots $s - a$, $s - b$, $s - c$.

13142. (Professor SCHWATT.)—Prove that, if through the centre of perspective D of a given triangle ABC and its BROCARD-triangle A′B′C′ be drawn straight lines so as to pass through S_a, S_b, S_c (S_a, S_b, S_c are the middle points of the sides BC, AC, and AB of the triangle ABC), and if S_aD_1 is made equal to DS_a, S_bD_2 equal to DS_b, S_cD_3 equal to DS_c, then (1) the figures $D_1O'AO$, $D_2O'BO$, and $D_3O'CO$ are parallelograms (O and O′ are BROCARD-points), (2) the triangles $D_1D_2D_3$ and ABC are equal, and (3) D_1A, D_2B, and D_3C intersect in S (S is the middle point of OO′).

Solution by Professors A. DROZ-FARNY, GOPALACHANAR, *and others.*



Si dans le plan d'un triangle de référence ABC, trois points P, Q, R ont respectivement pour coordonnées barycentriques, α, β, γ; β, γ, α; γ, α, β, on sait que le triangle PQR a même centre de gravité que ABC. Or les coordonnées bary-centriques de D sont $1/a^2$, $1/b^2$, $1/c^2$; celles de O, $1/b^2$, $1/c^2$, $1/a^2$; celles de O′, $1/c^2$, $1/a^2$, $1/b^2$. Le triangle DOO′ a donc même centre de gravité g que ABC. Soit S le milieu de OO′. Dans le triangle DgS_a, on a

$$\frac{Sg}{SD} \cdot \frac{AS_a}{Ag} \cdot \frac{D_1D}{D_1S_a} = \tfrac{1}{3} \cdot \tfrac{3}{2} \cdot \tfrac{2}{1} = 1.$$

Donc AD_1 passe par le point milieu S de OO′ et elle est divisée en ce point en parties égales; il en est de même des deux droites BD_2 et CD_3. Comme OS = O′S et AS = SD_1, la figure D_1OAO' est un parallélogramme, et enfin les points D_1, D_2, D_3 étant les symétriques de A, B, et C par rapport un point S, le triangle $D_1D_2D_3$ est égal au triangle ABC.

13244. (P. W. FLOOD.)—Given the hypotenuse of a right-angled triangle, to construct it, when the rectangle under the segments of one of the other sides made by the bisector of the angle opposite is equal to a given square or a maximum.

Solution by D. BIDDLE, *the* PROPOSER, *and others.*

Let the hypotenuse equal unity, and the side which forms with it the bisected angle equal x. Then the other side = $(1 - x^2)^{\frac{1}{2}}$, and the segments

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of it are $x(1-x^2)^{\frac{1}{2}}/(1+x)$ and $(1-x^2)^{\frac{1}{2}}/(1+x)$, their product being $x(1-x)/(1+x)$. If this $= a^2$, then $x = \frac{1}{2}\{1-a^2 \pm (1-6a^2+a^4)^{\frac{1}{2}}\}$. In order that $(1-6a^2+a^4)^{\frac{1}{2}}$ may be real, the maximum value of a^2 is $3-2\sqrt{2}$, in which case $x = \sqrt{2}-1$, and this same value is arrived at by taking the differential coefficient of $x(1-x)/(1+x)$, namely, $(1-2x-x^2)/(1+x)^2 = 0$. With these values before us, the construction is extremely simple in both cases if a semicircle be described upon the hypotenuse as base.

13318. (Professor BARISIEN.)—Dans un triangle ABC on considère les centres O_B et O_C des cercles ex-inscrits dans les angles B et C et on projette ces centres en O'_B et O'_C sur le côté BC. La droite O_CO_B rencontre BC en A'. Montrer que (1) les points A et A' sont conjugués harmoniques des points O_C et O_B; et (2) l'ellipse ayant pour sommets du grand axe les points O'_C et O'_B et passant par A jouit de la propriété d'être tangente à la droite O_CO_B et d'avoir pour foyers les points B et C.

Solution by T. SAVAGE, M.A.; W. E. JEFFARES, B.A.; *and others.*

A and A' are centres of similitude of the ex-circles, and therefore harmonic conjugates of their centres O_C and O_B.

Again

$$O'_CO'_B = O'_CB + BC + CO'_B,$$

$$O'_CB = CO'_B = s-a,$$

and $$BC = a;$$

$$\therefore\ O'_CO'_B = b+c.$$

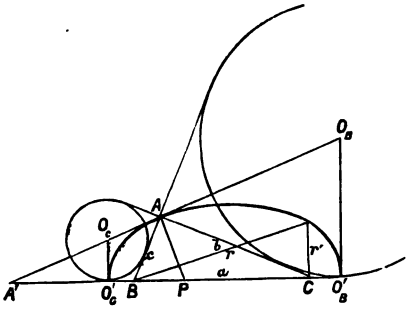
Hence the ellipse

$$r+r' = O'_CO'_B$$

passes through A. The normal AP bisects $\angle BAC$, and is perpendicular to O_CO_B, since

$$\angle O_CAB = \angle O_BAC.$$

Therefore O_CO_B is a tangent, B and C being foci.



13246. (Rev. T. WIGGINS, B.A.)—A movable line through the origin meets the fixed straight line $lx+my+n=0$ at P, and the fixed conic $ax^2+2hxy+by^2+2qx+2fy+c=0$ at Q and S. R is taken so that P and R divide QS harmonically. Prove that the locus of R may be written in the form $n(ax^2+2hxy+by^2+2qx+2fy+c) = (lx+my+n)(gx+fy+c)$.

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Solution by W. E. JEFFARES, B.A.; W. E. HEAL, M.A.; *and others.*

Let O be the origin, and $y = \tan\theta . x$ the transversal through O. The vectors OP, OQ, OR, OS are found from the equations

$$(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)\,r^2 + 2(g\cos\theta + f\sin\theta)\,r + c = 0,$$

$$(l\cos\theta + m\sin\theta)\,\mathrm{OP} + n = 0,$$

and

$$\frac{1}{\mathrm{PQ}} + \frac{1}{\mathrm{PS}} = \frac{2}{\mathrm{PR}}$$

(as QS is cut harmonically in P and R),

$$(\mathrm{PQ} + \mathrm{PS})\,\mathrm{PR} = 2\mathrm{PQ} \times \mathrm{PS},$$

$$(\mathrm{OQ} + \mathrm{OS} - 2\mathrm{OP})\,(\mathrm{OR} - \mathrm{OP}) = 2\,(\mathrm{OS} - \mathrm{OP})\,(\mathrm{OQ} - \mathrm{OP}),$$

or

$$(\mathrm{OQ} + \mathrm{OS} - 2\mathrm{OP})\,\mathrm{OR} = 2\mathrm{OS} \times \mathrm{OQ} - (\mathrm{OS} + \mathrm{OQ})\,\mathrm{OP}.$$

Now putting the values of OQ, OS, OP, got from the above equations, and ρ for OR, also clearing of fractions, we have the following equation: —

$$\rho\left\{n(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta) - (g\sin\theta + f\sin\theta)\,(l\cos\theta + m\sin\theta)\right\} = c\,(l\cos\theta + m\sin\theta) - (g\cos\theta + f\sin\theta)\,(l\cos\theta + m\sin\theta).$$

Multiply both sides by ρ, and, putting x for $\rho\cos\theta$ and y for $\rho\sin\theta$, we have

$$n(ax^2 + 2hxy + by^2) - (gx + fy)\,(lx + my) = c\,(lx + my) - (gx + fy)\,n,$$

or, as easily seen by adding $(gx + fy + c)\,n$ to both sides,

$$n(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) = (gx + fy + c)\,(lx + my + n),$$

required equation for the locus of R.

13210. (F. A. L. KITCHIN, B.A.)—Two stars whose elements are unknown rise together, and bear S. $\alpha°$ W. at the same moment. Show that the latitude (l) of the place is given by $\tan l = \sin\alpha$.

Solution by H. W. CURJEL, M.A.; C. BICKERDIKE; *and others.*

Let P be the pole. Draw great circle PN perpendicular to the horizon QN, and let small circle with pole P and radius $\frac{1}{2}\pi - l$ cut QN in Q. Then

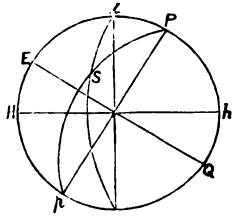
$$\angle \mathrm{PQN} = \alpha$$

and

$$\mathrm{PN} = l;$$

$$\therefore \frac{\sin(\frac{1}{2}\pi - l)}{\sin\frac{1}{2}l} = \frac{\sin l}{\sin\alpha};$$

$$\therefore \tan l = \sin\alpha.$$



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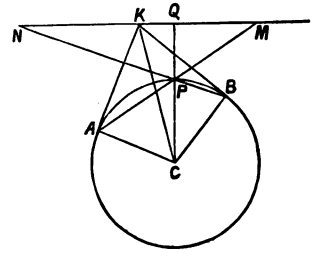
13177. (M. Brierley.)—From a given point draw two tangents to a given circle and join the points of contact. From the centre of the circle draw any radius (produced if necesary), perpendicular to which draw a line from the given point, and produce it indefinitely. If then lines be drawn from the points of contact of the tangents to intersect in the extremity of the radius before drawn, they will intercept on the indefinite line a portion of it of a constant magnitude.

Solution by Professors Radhakrishnan, Mukhopadhyay, *and others.*



Let $\angle ACB = 2L$, a constant,
and $\angle KCP = \theta$, a variable;
$\therefore$ $\angle APC = 90 - \frac{1}{2}(L+\theta)$,
and $\angle BPC = 90 - \frac{1}{2}(L-\theta)$.
Also $KC \cos L$ = the radius r;
$\therefore$ $PQ = CQ - CP = r \sec L \cos\theta - r$.

The area of the triangle MPN
$(= \frac{1}{2}MN \times PQ)$
$= \frac{1}{2}MP \,.\, NP \sin \angle MPN$,
i.e., $\propto MP \,.\, NP$,
i.e., $\propto PQ^2 \sec \angle MPQ \sec \angle NPQ$,
i.e., $\propto PQ^2/(\cos \angle APC \cos \angle BPC)$;

$\therefore$ $PQ/MN \propto \cos \angle APC \cos \angle BPC$, *i.e.*, $\propto (\cos\theta - \cos L)$.

But $PQ = (r/\cos L)(\cos\theta - \cos L)$, *i.e.*, $\propto (\cos\theta - \cos L)$.

Therefore MN is constant. [For another solution, see Vol. lxvi., p. 59.]

13287. (Editor.)—Prove that, in any triangle, we have, using the ordinary notation, $(r_a - r)(r_b - r)(r_c - r) = 4Rr^2$.

Solution by Rev. J. Cullen, B.A.; Rev. T. Wiggins, B.A.; *and others.*

The left-hand side may be written thus,

$$[S/(s-a) - S/s][S/(s-b) - S/s][S/(s-c) - S/s]$$
$$= Sabc/s^2 = 4RS^2/s^2 = 4Rr^2.$$

13283. (Professor Alev.)—Telegraph poles are a yards apart; for how many minutes must one count poles in order that the number of poles counted may be equal to the number of miles per hour that the train is running?

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Solution by D. BIDDLE, Rev. R. H. STEWART, *and others.*

Let x = the required number of minutes, y = the number of posts, and a = one-mth of a mile. Then 1760 yards = distance in $60/y$ minutes, $1760y/m$ = distance in x minutes; therefore $x = 60/m = 3a/88$.

13351. (Professor KRISHNAMACHARRY.) — If a weight P balance a weight W in that system of pulleys in which each pulley hangs by a separate string, show that, if P be changed to P′ and W to W′, P′ will descend with acceleration f such that

$$f\left\{2^{2n}\mathrm{P}' + \mathrm{W}' + \tfrac{1}{2}(2^n+1)(2^n\mathrm{P}-\mathrm{W})\right\} = 2^n g\left\{2^n(\mathrm{P}'-\mathrm{P}) + \mathrm{W}-\mathrm{W}'\right\},$$

all the pulleys being of equal weight.

Solution by H. W. CURJEL, M.A.; W. J. GREENSTREET, M.A.; *and others.*

$2^n\mathrm{P} = \mathrm{W} + w(2^n-1)$, where w = weight of each pulley.

If $\dot{x}$ is the velocity of P′ when it has descended a distance x, the equation of work and energy gives

$$\left\{(\mathrm{P}'-\mathrm{P})x - \frac{\mathrm{W}'-\mathrm{W}}{2^n}x\right\}g = \tfrac{1}{2}\dot{x}^2\left\{\mathrm{P}' + \frac{\mathrm{W}'}{2^{2n}} + \frac{w}{3}\left(1-\frac{1}{2^{2n}}\right)\right\},$$

for the work done = 0, when P and W are the weights.

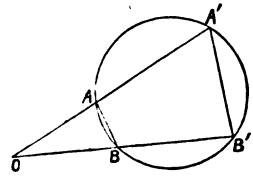
$$\therefore\quad f = \ddot{x} = \left\{\mathrm{P}'-\mathrm{P}-\frac{\mathrm{W}'-\mathrm{W}}{2^n}\right\} \Big/ \left\{\mathrm{P}' + \frac{\mathrm{W}'}{2^{2n}} + \frac{w(2^{2n}-1)}{3\times 2^{2n}}\right\}.$$

$$\therefore\quad \left\{2^{2n}\mathrm{P}' + \mathrm{W}' + \tfrac{1}{2}(2^n+1)(2^n\mathrm{P}-\mathrm{W})\right\} = 2^n g\left\{2^n(\mathrm{P}'-\mathrm{P}) + \mathrm{W}-\mathrm{W}'\right\}.$$

13230. (Professor CAZAMIAN.)—Soient O le centre d'inversion, A et B deux points d'une figure, A′ et B′ les points correspondants de la figure inverse. Démontrer que la médiane et la symédiane du triangle AOB, issues de O, sont respectivement symédiane et médiane du triangle A′OB′.

Solution by H. W. CURJEL, M.A.; W. E. JEFFARES, B.A.; *and others.*

A, B, A′, B′ are concyclic; therefore triangles OAB, OB′A′ are equiangular, and hence the medians of triangles OAB, OB′A′ through O coincide with the symmedians of triangles OB′A′, OAB through O, respectively.



13248. (M. BRIERLEY.)—A straight line AB and a point P are given in position; through P another line is drawn to intersect AB in B, and PB is so divided in C that PC, PB is given in magnitude and a constant quantity; required the locus of C.

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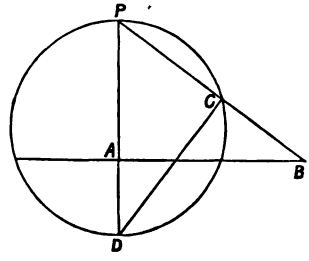
Solution by W. E. JEFFARES, B.A.; Prof. RADHAKRISHNAN; *and others.*

From P let fall the perpendicular PA on LL', and produce PA to D, till PD × PA = the given constant quantity. On PD as diameter, describe a circle; this circle will be the locus of C. For draw any line PB. Join DC. ∠BCD is right, and ∠BAD is also right. Therefore the points A, B, C, D are concyclic, and the rects. PA × PD = PB × PC. Therefore the circle PCD is the required locus.



13306. (Professor SANJÁNA.)—In an isosceles triangle ABC, right-angled at C, CA is bisected in D, and DG is drawn perpendicular to AB; prove that AG = ¼GB. Hence, or otherwise, inscribe (geometrically) in a quadrant of a circle a square having two angular points on the arc. [The first part proves that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{3} = \frac{1}{4}\pi$.]

Solution by J. W. WEST, M.A.; W. E. JEFFARES, B.A.; *and others.*

Let ABC be the isosceles triangle, right-angled at (C), and D the bisecting point of (AC).

Then, as ∠(AGD) is a right angle by construction,

∠(ADG) = ∠(ABC)

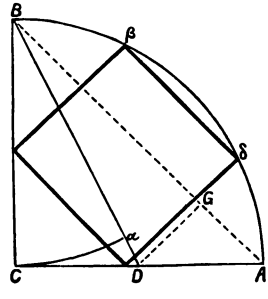
[Euc. I. 32 and Ax. 3]

and DG : DA (= ½AC = ½√2 AB)

:: AC (= ½√2 AB) : AB;

∴ DG = ¼AB = ¼GB.

Hence, to inscribe a square in the quadrantal area (ABC), having two angular points on the arc (AB).



Bisect (AC) at (D), and draw the straight line (BD).

From the centre (B) at the distance (BC) describe the arc (Cα).

From the centres (A) and (B) at the distance (Cα), strike arcs intersecting arc (AB) in (δ) and (β).

(δβ) shall be one side of the inscribed square.

Because arc (δβ) = arc (AB) − 2 arc (Cα)

= by construction, $\frac{1}{2}\pi - 2\tan^{-1}\frac{1}{2} = 2\tan^{-1}\frac{1}{3}$.

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13331. (Rev. W. A. WHITWORTH, M.A.)—If n chords be drawn at random in a circle, show that the chances of

$\frac{1}{2}n(n-1)$ intersections $= N$, $\frac{1}{2}n(n-1)-1$ intersections $= nN$,

no intersections $= 2^n \div (n+1)!$, and one $= 2^n n(n-1) \div (n+2)!$;

where $N = 2^n n!/(2n)! = 1 \div$ product of the first n odd numbers.

Solution by H. FORTEY.

1. Preliminary, and notation. A random chord in a circle is the join of any two random points on the circumference, and n random chords result from joining in any way in pairs $2n$ random points. Also, the points being numbered in succession 1, 2, ..., $2n$ as we go round the circle, the number of cuts of the n chords depends in no degree on the absolute position of the random points, but entirely on their pairs of points (as indicated by their numbers) which constitute the chords. A little consideration will show that $2n$ points can be joined by n chords in $\psi(n)$ ways, where $\psi(n) = 1.3.5\ldots(2n-1) = \frac{(2n)!}{2^n.n!}$.

Let n_x denote the number of ways in which $2n$ points can be joined by n chords making x cuts, and let $p_x(n)$ be the chance that the chords are so drawn. Then $p_x(n) = n_x \div \psi(n)$.

2. To find n_0. Take $2n+2$ points and join them in pairs by $n+1$ chords, making no cuts, and let PQ be any chord issuing from P, which suppose fixed. Then Q may range from one point to another, subject only to the condition that no chord crosses PQ, which means that, of the remaining n chords, r are on one side of PQ and $n-r$ on the other, where r varies from 0 to n. Moreover, the r chords and the $n-r$ chords form two distinct groups separated by PQ, and there must be no cut of either group.

$$\therefore \quad (n+1)_0 = \sum_{r=0}^{r=n} (n-r)_0 . r_0,$$

or, since $0_0 = 1_0 = 1$, as there can be no cut where there are no chords or only one chord,

$$(n+1)_0 = n_0 + (n-1)_0 + (n-2)_0 . 2_0 + \ldots + 2_0 . (n-2)_0 + (n-1)_0 + n_0 \ldots (a).$$

Now let $\phi_0(x)$ be the generating function of n_0, so that

$$\phi(x) = 1 + x + 2_0 . x^2 + 3_0 . x^3 + \ldots + n_0 . x^n + \ldots .$$

Then $(\phi_0 x)^2$ generates $n_0 + (n-1)_0 + (n-2)_0 . 2_0 + \ldots = (n+1)_0$ by (a).

But $\frac{\phi_0(x)-1}{x}$ also generates $(n+1)_0$;

$$\therefore \quad (\phi_0 x)^2 = \frac{\phi_0(x)-1}{x}, \quad \text{whence} \quad \phi_0(x) = \frac{1}{2x}\left\{1-(1-4x)^{\frac{1}{2}}\right\}.$$

And $$(1-4x)^{\frac{1}{2}} = 1 - 2x - 2x^2 - 4x^3 - \ldots - \frac{2(n)!}{n!(n+1)!}x^{n+1} + \ldots;$$

$$\therefore \quad \phi_0(x) = 1 + x + 2x^2 + \ldots + \frac{(2n)!}{n!(n+1)!}x^n + \ldots;$$

$$\therefore \quad n_0 = \frac{(2n)!}{n!(n+1)!} \quad \text{and} \quad p_0(n) = n_0 \div \psi(n) = \frac{2^n}{(n+1)!}.$$

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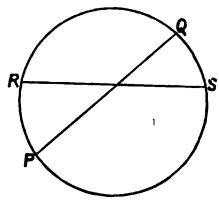
3. We have seen above that $2n$ points can be joined by n chords with no cuts in n_0 or $(n+1)_0$ ways, according as the chords form one group or two separate groups. In the next investigation, we shall require to know the number of ways in which these n chords can be drawn with no cuts when they form four separate groups.

The number of ways is evidently got by summing $(n-r)_0.(r-s)_0.(s-t)_0.t_0$ with the limits 0, s of t; 0, r of s; 0, n of r; and, omitting the work, which is easy enough, I may say that the result is $(n+3)_0-2(n+2)_0$.

4. To find n_1. If the chords make one cut, the chord PQ may or may not be cut by another chord.

Case I.—Let PQ be cut by RS. Then, if there are $n+2$ chords altogether, there are n still to be disposed of in four groups with no cuts, and the number of ways is

$$(n+3)_0-2(n+2)_0.$$



But, if there are n chords *altogether*, the result is $(n+1)_0-2n_0$, and the generator of this is $\frac{1-2x}{x}\phi_0(x)$.

Case II.—If no chord cuts PQ, suppose there are $n+1$ chords altogether; then, of the n disposable chords, $n-r$ may be on one side of PQ and r on the other, and there may be one cut of the $n-r$ and no cut of the r, and the $n-r$ may be on *either* side of PQ. Therefore the number of ways

$$= 2\sum_{r=0}^{r=n}(n-r)_1 . r_0,$$

and, since $0_1 = 1_1 = 0$,

$$= 2\{n_1 + (n-1)_1 . 1_0 + (n-2)_1 . 2_0 + \ldots + 2_1(n-2)_0\} = 2g(n), \text{ suppose.}$$

Now let $\phi_1(x)$ generate n_1, so that

$$\phi_1(x) = 2_1x^2 + 3_1x^3 + \ldots + n_1x^n + \ldots;$$

then, since $\phi_0(x) = 1 + x + 2_0x^2 + \ldots + n_0x^n + \ldots,$

we see that $2g(n)$ is generated by $2\phi_0(x) . \phi_1(x)$; and, if there are n chords altogether, the number of ways is $2g(n-1)$, and this is generated by $2x\phi_0(x) . \phi_1(x)$.

The results of the two cases added together make up n_1 or

$$n_1 = 2g(n-1) + (n+1)_0 - 2n_0.$$

Turning therefore to the generating functions, it would seem that

$$\phi_1(x) = 2x\phi_0(x)\,\phi_1 x + \frac{1-2x}{x}\phi_0(x) \text{ or } \phi_1(x)\{1-2x\phi_0(x)\} = \frac{1-2x}{x}\phi_0(x).$$

But, as there are no powers of x below x^2 on the left of the last equation, we must subtract such powers on the right. Making this correction, and remembering that $1-2x\phi_0(x) = (1-4x)^{\frac{1}{2}}$, we have

$$\phi_1(x)(1-4x)^{\frac{1}{2}} = \frac{1-2x}{x}\phi_0(x) - \frac{1-x}{x} = \frac{1-4x+2x^2}{2x^2} - \frac{1-2x}{2x^2}(1-4x)^{\frac{1}{2}}.$$

$$\therefore \quad \phi_1(x) = \frac{1-4x+2x^2}{2x^2}(1-4x)^{-\frac{1}{2}} - \frac{1-2x}{2x^2}.$$

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Now $$(1-4x)^{-\frac{1}{2}} = 1 + A_1 x + A_2 x^2 + \ldots + A_n x^n + \ldots,$$

where $$A_n = \frac{(2n)!}{(n!)^2},$$

and, since n_1 is generated by $\phi_1(x)$,

$$n_1 = A_n - 2A_{n+1} + \tfrac{1}{2}A_{n+2} = \frac{(2n)!}{(n-2)!\,(n+2)!} \text{ by reduction,}$$

and $$p_1(n) = n_1 \div \psi(n) = n(n-1)\frac{2^n}{(n+2)!}.$$

5. By a similar but longer investigation, I have determined the number of ways (and therefore the chance) of two cuts of n random chords:

$$n_2 = \tfrac{1}{2}\frac{(2n)!}{(n-3)!\,(n+2)!}, \qquad p_2(n) = n(n-1)(n-2)\frac{2^{n+1}}{(n+2)!}.$$

6. To find n_m, where $m = \frac{1}{2}n(n-1)$. To make m cuts, each chord must cut every other; and this is only possible if the chords are $1, n+1$; $2, n+2$; $\ldots$; $n, 2n$. Therefore this is the only way in which the chords can be drawn.

$$\therefore \quad n_m = 1, \quad p_m(n) = 1 \div \psi(n) = \frac{2^n . n!}{(2n)!}.$$

7. To find n_{m-1}. The number of ways of $m-1$ cuts is the same as the number of ways of diminishing by one the number of cuts in the diagram with m cuts. This may be done by uncrossing any two adjacent chords, leaving the rest untouched; and, as there are n pairs of adjacent chords,

$$n_{m-1} = n \quad \text{and} \quad p_{m-1}(n) = n \div \psi(n) = n \cdot \frac{2^n . n!}{(2n)!}.$$

8. In addition to the above results, I may add that, if $r \not> n-1$,

$$n_{m-r} = \frac{n \,.\, \overline{n+1 \ldots n+r-1}}{r!}, \quad p_{m-r}(n) = 2^n . n . \frac{(n+r-1)!}{r!\,(2n)!}.$$

Note on Question 8802. By D. Biddle.

Mr. Simmons, having seen fit, in Vol. lxv., p. 40, to bring to light again, after ten years, what he supposes to be an error on my part in Vol. xlv., p. 113, I venture to make the following additional remarks.

What I said at the beginning of the impugned passage was: "If p denote the chance of A. travelling with one or other of two ladies whose chances of travelling with him are equal, $\frac{1}{2}p$ will denote his chance of travelling with one in particular." Mr. Simmons himself had previously said: "If p denote the chance that A. travels with a lady." Surely, it is an outrage on masculine sentiment to suppose that no idea of "mutual exclusion" on the part of the two ladies entered A.'s mind. Now, if n be the number of compartments in the train, and each lady have an equal chance of occupying any one of them, $(n-1)/n^2$ will be A.'s chance of travelling with one lady in particular to the exclusion of the other, and $2(n-1)/n^2$ will be his chance of travelling with one or other, in that exclusive manner; the latter being p, the former is $\frac{1}{2}p$.

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But, even supposing "mutual exclusion" to be excluded, I contend that $\frac{1}{2}p$ was near enough in the particular instance. Accurately, the chance is $\frac{1}{2}p + 1/(2n^2)$, so that, even in Mr. SIMMONS's favourite train of three compartments only, the chance would be $\frac{1}{2}p + \frac{1}{18}$, and in an ordinary suburban train (of, say, sixty compartments) it would be $\frac{1}{2}p + \cdot00014$. Moreover, I contend that, if $\frac{1}{2}p$ and $\frac{1}{2}q$ be A.'s and B.'s chances of travelling with one of the two ladies in particular, which I have now shown to be sufficiently accurate for practical purposes, then the chance *that A. travels with one and B. with the other* is given not by pq, but by $2(\frac{1}{2}pq)$, as I previously stated in the passage referred to, the slight increase in the estimate required by accuracy (in the absence of "mutual exclusion") being practically negligible.

Of course, all four of the specified passengers may be in one compartment: the chance of which is possibly $1/n^3$ and ordinarily negligible.

If 60 compartments be allowed, the difference in the practical and accurate results is only one-eighth of that found in Professor FINKEL's Question 13080, on p. 90 of Vol. LXV.

I am satisfied if Mr. SIMMONS admits that the chance in question is nearer $\frac{1}{2}pq$ than pq. The former is my estimate, the latter his.

[Mr. SIMMONS replies as follows:—"If the $\frac{1}{2}p$ formula is unsound, as was clearly proved in Vol. LXV., surely there is an end of it. Mr. BIDDLE seems here to maintain that, provided it arrives at approximately accurate arithmetical results, incorrect reasoning in mathematics is nearly as good as correct reasoning. I entirely differ from him."

To which Mr. BIDDLE rejoins that correct reasoning does not lead us farther astray than incorrect.

13301. (R. CHARTRES.)—If n points be taken at random, one in each side of a regular polygon of n sides, prove that the mean value of the inscribed figure $= \cos^2(\pi/n)$, if the area of the polygon be unity.

Solution by H. W. CURJEL, M.A.; Professor MUKHOPADHYAY; *and others.*

If all the points except P remain fixed, the mean area of the inscribed polygon clearly equals area of polygon when P is at the middle point of the side in which it lies. Therefore the mean value of the inscribed polygon equals area of the polygon formed by joining the middle points of the successive sides of the given polygon $= \cos^2(\pi/n)$.

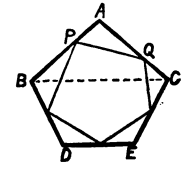
[The PROPOSER gives this solution:—

Let each side of the polygon be unity; then, since the mean value of $\triangle PAQ = \frac{1}{4}BAC$,

$\therefore$ mean value of inscribed figure

$$= \frac{n}{4} \cdot \tan\frac{(n-2)\pi}{2n} - \frac{n}{8} \cdot \sin\frac{(n-2)\pi}{n}.$$

Therefore, dividing by the area of the polygon, we get $\cos^2(\pi/n)$.]



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4815. (Dr. ARTEMAS MARTIN.)—If four dice be piled up at random on a horizontal plane, find the probability that the pile will not fall down.

Solution by H. W. CURJEL, M.A.

If the upper part of the pile will stand when the lowest die of that part is held firm, the range for equilibrium when it is placed on a die which is held firm is a^2 (the edges of the dice being a). The total range is $2a^2(1+\cos\theta+\sin\theta)$, where θ is the acute angle ($\not> \frac{1}{4}\pi$) between two edges, one of the lowest die of the upper part and the other of that on which it rests. Therefore, chance that the upper part will stand when placed on a die which is held firm

$$= \int_0^{\frac{1}{4}\pi} \frac{d\theta}{2(1+\cos\theta+\sin\theta)} \Big/ \int_0^{\frac{1}{4}\pi} d\theta = \frac{\log 2}{\pi}.$$

Hence, as in my solution to Question 10905 (*Reprint*, Vol. LXI., p. 114), the chance that the pile will stand when complete $= \left(\frac{\log 2}{\pi}\right)^3$; and, in case of n dice, chance $= \left(\frac{\log 2}{\pi}\right)^{n-1}$.

In the above solution, I have assumed that the pile may be supported till complete. This view makes the question a much simpler one, but is probably not that intended by Dr. MARTIN, as the same remarks would apply to its interpretation as to that of Questions 5263 and 10905 (see Vol. LXII.).

13271. (R. KNOWLES, B.A.)—If p_r denote the coefficient of x_r in the expansion of $(1+x)^{-\frac{1}{2}}$, prove that

$$p_1 - \tfrac{1}{2}p_2 + \tfrac{1}{3}p_3 - \&c.\ \text{ad infin.} \equiv 2\log\tfrac{1}{2}(1+2^{\frac{1}{2}}).$$

Solution by Rev. J. CULLEN, B.A.; Professor SANJÁNA; *and others.*

The coefficient of x^r in $(1-x)^{-\frac{1}{2}}$ is

$$1.3.5.7\ldots(2r-1)/2^r r! = \frac{2}{\pi}\int_0^{\frac{1}{4}\pi} \sin^{2r}\theta d\theta;$$

$$\therefore\quad S = \frac{2}{\pi}\int_0^{\frac{1}{4}\pi} \sum_{r=\infty}^{r=1} (-1)^{r+1} \sin_{2r}\theta\, d\theta/r = \frac{2}{\pi}\int_0^{\frac{1}{4}\pi} \log(1+\sin^2\theta)\, d\theta$$

$$= 2\log\tfrac{1}{2}(1+2^{\frac{1}{2}}).$$

[Similarly, $(1+x)^{-\frac{1}{2}}$ gives

$$S = \frac{2}{\pi}\int_0^{\frac{1}{4}\pi} \log(1-\sin^2\theta)\, d\theta = \frac{4}{\pi}\int_0^{\frac{1}{4}\pi} \log\cos\theta\, d\theta = 2\log\tfrac{1}{2}.]$$

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Note on Question 13066. By H. W. CURJEL, M.A.

From the PROPOSER's remark on my solution to this question (Vol. LXV., p. 95), it seems that it is not sufficiently clear how the second of the two component events may be reduced to one independent of the first. This may be done by considering the positions of B in pairs B_1, B_2, such that B_1B_2 is a diameter, or such that $\angle B_1OA + \angle B_2OA = \pi$, O being the centre of the circle. A slight modification of the proof applies to the following:—If three random points are taken in an area (not necessarily consisting of one piece) which is such that equal areas lie on the two sides of any straight line through a fixed point O, the chance that a line can be drawn through O such that all three points lie on the same side of it is $\frac{3}{4}$.

13266. (J. J. BARNIVILLE, B.A.)—Sum the series

$$\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \frac{8}{1+x^8} + \dots .$$

Solution by Rev. J. CULLEN, B.A.; H. W. CURJEL, M.A.; *and others.*

We have $$\frac{1}{1+x} + \frac{2}{1+x^2} = \frac{x^2+2x+3}{x^3+x^2+x+1},$$

$$\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} = \frac{x^6+2x^5+3x^4+4x^3+5x^2+6x+7}{x^7+x^6+x^5+x^4+x^3+x^2+x+1};$$

$$\therefore S = x^{2^n-2} \sum_{r=2^n-1}^{r=1} r/x^{r-1} \Big/ \sum_{r=2^n-1}^{r=0} x^r = \left[(x^{2^n}-x)/(x-1) - 2^n + 1\right]/(x^{2^n}-1).$$

13270. (J. W. WEST.)—Evaluate dy/dx on the supposition that $\int ds$ is an assigned magnitude, when $\int y\,dx$ (immersed in a fluid) gives a maximum moment of total pressure.

Solution by Rev. J. CULLEN, B.A.; *the* PROPOSER; *and others.*

The moment of the total pressure is $\int x^2y\,dx$, and

$$\delta\left\{\int x^2y\,dx + a\int ds\right\} = \int \delta\left\{x^2y\,dx + a\,ds\right\}$$

$$= \int\left\{2xy\,dx\,\delta x + x^2dx\,\delta y + x^2y\,\delta dx + \frac{a\,dy}{ds}\,\delta dy + \frac{a\,dx}{ds}\,\delta dx\right\};$$

therefore $x^2dx - ad(dy/ds) = 0$, and $dy/dx = \dfrac{x^3}{\sqrt{(c^6-x^6)}}$,

which is the equation to a curve very much like a wine glass without the stem.

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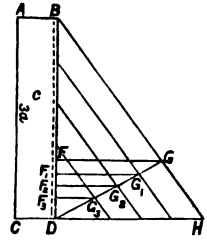
13049. (Professor ANTHONY.)—A vertical slit is made in the middle of the side of a rectangular box containing water. Find the time required to empty the box.

Solution by C. BICKERDIKE, Professor SARKAR, *and others.*

Let ABC represent a side of the box, and let the ordinate DH represent the force of the water at its bottom; then, as there is no force at the top AB, the triangle DHB will evidently represent the whole of the forces from B to D. Again, draw FG to pass through the centroid of the triangle; then the centre of forces is represented by FG, which is two-thirds from B.



Calc. BD = $3a$, area of slit = a',
contents of box = e, time in emptying = t,
velocity at F = v;

then $v = (4ga)^{\frac{1}{2}} = 2(ga)^{\frac{1}{2}}$;

$\therefore \quad 2a't(ga)^{\frac{1}{2}} = e; \qquad \therefore \quad t = e/\{2a'(ga)^{\frac{1}{2}}\}.$

[A correspondent cannot see that this takes into account that the water gets lower in the box and FG less. Join DG.]

13251. (Professor IGNACIO BEYENS.)—A uniform rigid rod, of length $2a$, can turn in a horizontal plane about its middle point. At one end a string is tied which passes over a fixed pulley, vertically over that end, and at a distance b from it, and is then fastened to a given weight. The rod is then turned through an angle θ, and kept at rest in that position by a horizontal force P perpendicular to the rod through its other end. Prove that P will be a maximum if $\tan^4 \frac{1}{2}\theta = b^2/(b^2 + 4a^2)$, where a is the radius of the circle circumscribing the polygon, r the distance of P from its centre O, and θ the angle OP makes with the radius drawn to any vertex of the polygon.

Solution by H. W. CURJEL, M.A.; Prof. RADHAKRISHNAN; *and others.*

Taking moments about the centre of the rod,

$$P = \frac{2Wa \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}{(4a^2 \sin^2 \frac{1}{2}\theta + b^2)^{\frac{1}{2}}}.$$

Therefore, for a critical value of P,

$$(4a^2 \sin^2 \tfrac{1}{2}\theta + b^2)(\cos^2 \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta) = 4a^2 \sin^2 \tfrac{1}{2}\theta \cos^2 \tfrac{1}{2}\theta;$$

$$\therefore \quad -4a^2 \sin^4 \tfrac{1}{2}\theta + b^2 \cos^4 \tfrac{1}{2}\theta - b^2 \sin^4 \tfrac{1}{2}\theta = 0;$$

$$\therefore \quad \tan^4 \tfrac{1}{2}\theta = b^2/(4a^2 + b^2).$$

This value is easily seen to make P a maximum.

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13316. (Professor WILKES.)—Let n be any number, and let $n^3 + 1 = x$; prove that $x^3 + (2x-3)^3 + (nx-3n)^3 = n^3x^3$.

Solution by T. SAVAGE, M.A.; J. W. WEST; *and others.*

$$x^3 + (2x-3)^3 = (x + 2x - 3)(x^2 - 2x^2 + 3x + 4x^2 - 12x + 9)$$
$$\equiv 3(x-1)(3x^2 - 9x + 9) = 3n^3(3x^2 - 9x + 9);$$
$$(nx-3n)^3 = n^3(x^3 - 9x^2 + 27x - 27).$$
$$\therefore\quad x^3 + (2x-3)^3 + (nx-3n)^3 = n^3x^3.$$

13294. (C. BICKERDIKE.) — Assuming that Venus and the Earth move in circular orbits with the Sun at the centre, what is the angle of position of the former when at its greatest brilliancy, the intensity of light varying inversely as the square of the distance?

Solution by Rev. J. CULLEN, B.A.; Professor GOPALACHANAR; *and others.*

If E be the elongation of Venus, we have

$$\cos^2 E + \frac{4}{3}\frac{r}{a}\cos E = \frac{4}{3} \quad \text{(GODFRAY'S } \textit{Astronomy}\text{, p. 288).}$$

Taking a, the Earth's distance from the Sun, as unity, r, that of Venus, is ·72333; $\cos E = ·76913$. $\therefore\ E = 39° 43'$, nearly.

13291. (H. W. CURJEL, M.A. Suggested by Quest. 13258.)—If $x = 0$ is the base of a cone, vertex P, which envelopes a sphere $x^2 + y^2 + z^2 = 2ax$, show that locus of P (1) when the area of the base of the cone is constant is $(x^2 + y^2 + z^2 - 2ax)\,x = m(x-2a)^3$; (2) when the volume of the cone is constant is $(x^2 + y^2 + z^2 - 2ax)\,x^3 = n(x-2a)^3$.

Solution by Rev. J. CULLEN, B.A.; Prof. RAMASWAMI AIYAR; *and others.*

The enveloping cone having (x_1, y_1, z_1) for vertex is

$$(\Sigma x^2 - 2ax)(\Sigma x_1^2 - 2ax_1) - \{\Sigma x x_1 - a(x + x_1)\}^2 = 0.$$

Putting $x = 0$, we get the conic

$$y^2(x_1^2 + z_1^2 - 2ax_1) - 2yzy_1z_1 + z^2(x_1^2 + y_1^2 - 2ax_1) + 2ayx_1y_1 + 2azx_1z_1 - a^2x_1^2 = 0 \quad \ldots\ldots\ldots\ldots (1).$$

Adopting ordinary notation, we have the area of (1)

$$= \pi\Delta(d\Delta/dc)^{-\frac{1}{2}} = \pi a^2 x_1^2 \{\Sigma x_1^2 - 2ax_1\}^2 / \{x_1(x_1 - 2a)(\Sigma x_1^2 - 2ax_1)\}^{\frac{1}{2}} = k;$$

$\therefore\ (\Sigma x_1^2 - 2ax_1)\,x_1 = m(x_1 - 2a)^3$, where $m^{\frac{1}{3}} = k/\pi a^2 = $ constant.

(2) If V be the volume, then $V = kx_1/3$; hence

$$(\Sigma x_1^2 - 2ax_1)\,x_1^3 = n(x_1 - 2a)^3 \quad (n^{\frac{1}{3}} = 3V/\pi a^2 = \text{constant}).$$

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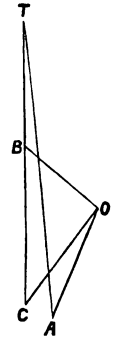
13293. (W. J. Dobbs, M.A.)—Four heavy uniform rods of the same thickness and material, and of lengths a, b, c, d, are freely jointed at their extremities to form a quadrilateral framework. A fine string of given length connects the hinges ab and cd. The framework rests in a vertical plane with the rod d held in a given position. Prove the following method for determining the tension of the string. Draw TBC vertically downwards making $TB = \frac{1}{2}(a+b)$ and $BC = \frac{1}{2}(b+c)$. Let the straight lines through B, C parallel to the rods b, c respectively meet in O. Draw TA parallel to the string and OA parallel to the rod a to meet TA in A. Then AT represents the length of a rod whose weight is equal to the tension of the string.

Solution by Rev. J. Cullen, B.A.; Professor Swaminatha; *and others.*



Taking the unit of weight to be the weight of a unit length, then the weight of $a = a$, and it may be replaced by $\frac{1}{2}a$ acting at each extremity. Similarly for b and c. Hence the vertical force acting at the hinge ab is $\frac{1}{2}(a+b)$ and $\frac{1}{2}(b+c)$ at bc.

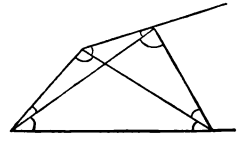
Laying off $BC = \frac{1}{2}(b+c)$ on a vertical line, and drawing OB, OC parallel to b and c, we get the reactions along these rods; and, as the hinge ab is in equilibrium by $\frac{1}{2}(a+b)$, by the reactions along a and b, and by the tension of the string, the value of the latter is given by completing the force diagram as stated.

13329. (G. Heppel, M.A.)—Find the least possible value of the acute angle between the diagonals of a quadrilateral described about a circle.

Solution by H. W. Curjel, M.A.; Prof. Krishnamacharry; *and others.*

If we consider a quadrilateral described about a circle with two opposite sides parallel and the other two equal but not parallel, then, by making one of the parallel sides very small, we can clearly make the angle between the diagonals as small as we please. If the circle is escribed to the quadrilateral, we can make the angle between the internal diagonals as small as we please by moving one side close up to the opposite side.

[What the Proposer really meant by his question was: "such that every angle between a diagonal and a side is acute," as shown by him in the accompanying diagram.]



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13375. (Professor SYLVESTER, Athenæum Club, 18*th Dec.*, 1896.)—

If $$\frac{Fx}{(1-x)^n} = 1 + 2^{n-1}x + 3^{n-1}x^2 + \&c.,$$

prove that Fx is a symmetrical function of the degree $(n-2)$ in x, the sum of whose coefficients is the factorial of $(n-1)$.

Ex.—Let $n = 5$. Call $Fx = 1 + Bx + Cx^2 + Dx^3$; by the rule given above, $D = 1$, $C = B$, and $1 + B = 12$, so that $B = 11$, $C = 11$, $D = 1$, and accordingly

$$\frac{1 + 11x + 11x^2 + x^3}{(1-x)^5} = 1 + 16x + 81x^2 + 256x^3 + \dots .$$

Let $n = 6$. Call $Fx = 1 + Bx + Cx^2 + Dx^3 + Ex^4$. Then obviously $B = 32 - 6 = 26$, and, by our rule, $D = 26$, $E = 1$, and $C = 120 - 54 = 66$. So, if $n = 7$, if $Fx = 1 + Bx + Cx^2 + Cx^3 + Bx^4 + x^5$, we have $B = 64 - 7 = 57$, $C = 1\frac{2}{3}\frac{6}{2} - 58 = 302$, and it will be found that

$$\frac{1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5}{(1-x)^7} = 1 + 64x^2 + 729x^3 + 4096x^4 + \dots .$$

Thus, with scarcely any work of computation, we may find the following table of coefficients in Fx corresponding to all values of n from 1 to 9 inclusive, viz.: 1; 1; 1, 1; 1, 4, 1; 1, 11, 11, 1; 1, 26, 66, 26, 1; 1, 57, 302, 302, 57, 1; 1, 120, 1312, 2416, 1312, 120, 1; 1, 247, 3693, 16219, 16219, 3693, 247, 1.

Solution by R. KNOWLES, B.A.; Professor STEGGALL; *and others.*

Let $S = 1 + 2^n x + 3^n x^2 + \dots$; then, using the symbolical method,

$$S = (D+1)^n u, \quad \text{if} \quad u = \frac{1}{1-x}.$$

If $$n = 1, \quad S = \left(x\frac{d}{dx} + 1\right)u = \frac{1}{(1-x)^2} \quad \dots\dots\dots\dots\dots\dots (1);$$

$$n = 2, \quad S = (D+1)^2 u = (D^2 + 2D + 1)u = \left\{D(D-1) + 3D + 1\right\}u$$

$$= \left(x^2\frac{d^2}{dx^2} + 3x\frac{d}{dx} + 1\right)u = 2x^2u^3 + 3xu^2 + u = \frac{1+x}{(1-x)^3} \quad \dots (2);$$

$$n = 3, \quad S = (D+1)^3 u = \left(x^3\frac{d^3}{dx^3} + 6x^2\frac{d^2}{dx^2} + 7x\frac{d}{dx} + 1\right)u$$

$$= 6x^3u^4 + 12x^2u^3 + 7xu^2 + u = \frac{1 + 4x + x^2}{(1-x)^4}.$$

The sum of the coefficients is found by putting $x = 1$ in $F(x)$, and then all the terms of the differentiation vanish except the first, and the sum is the factorial of $n-1$.

4864. (Professor CAVALLIN.)—Two points P and Q are taken at random on the area of a vertical circle; find the probability that the time of descent for a particle down the straight line PQ, from P to Q, may be less than that from P down the straight line of quickest descent to the circle.

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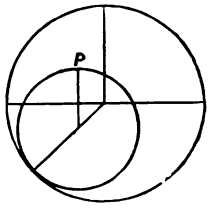
Solution by H. W. CURJEL, M.A.



Take the equation to the circle $x^2+y^2=a^2$, the axis of x being drawn vertically upwards. Radius of circle, touching the given circle, of which P (x, y) is the highest point, is

$$\frac{a^2-x^2-y^2}{2(a-x)},$$

and if condition is satisfied Q must be in this circle; therefore chance required

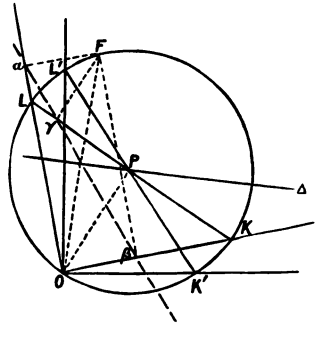
$$= \frac{1}{\pi a^2}\int_{-a}^{a}\int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} \frac{1}{a^2}\left\{\frac{a^2-x^2-y^2}{2(a-x)}\right\}^2 dx\,dy$$

$$= \frac{4}{15\pi a^4}\int_{-a}^{a}(a+x)^2\sqrt{(a^2-x^2)}\,dx = \frac{4}{15\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(1+\sin\theta)^2\cos^2\theta\,d\theta$$

$$= \frac{4}{15\pi}\times\frac{5\pi}{8} = \frac{1}{6}, \quad \text{where } x = a\sin\theta.$$

13340. (C. E. HILLYER.)—O is a fixed point, P any point in a given straight line, and PK is drawn equally inclined with PO to a fixed straight line through O. Prove that PK always touches a fixed parabola; and, further, that if PK′ be drawn equally inclined with PO to a second fixed straight line through O, then each side of the triangle PKK′ touches one of three fixed confocal parabolas.

Solution by Professor A. DROZ-FARNY; H. W. CURJEL, M.A.; *and others.*



Soit Δ la droite donnée; élevons en O la perpendiculaire OL sur OK. La circonférence décrite de P comme centre avec PO comme rayon coupe les droites OK et OL respectivement en K et L et KL sera la tangente cherchée.

Soit F le point symétrique de O par rapport à Δ; le triangle LOK étant inscrit dans la circonférence, les pieds α, β, γ des perpendiculaires abaissées de F sur OL, OK, KL sont en ligne droite, la droite de SIMSON de F. α et β étant deux points fixes, il en résulte que PK enveloppe une parabole ayant F comme foyer et $\alpha\beta$ comme tangente au sommet.

De la même manière PK′ enveloppera une parabole ayant F comme foyer et sa droite de SIMSON par rapport au triangle L′OK′ comme tangente au sommet. Enfin angle KFK′ = KOK′ = constant; donc KK′ enveloppe aussi une parabole de foyer F.

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615. (W. PRIESTLEY.)—Suppose a person standing in a railway carriage, which moves uniformly at the rate of 25 miles per hour, were to drop a ball to the bottom of the carriage, a height of 5 feet. Find the length of the track described by the ball in its descent.

Solution by Professor ZERR, M.A.

The equation to the curve described by the ball is, since the angle of projection is zero $y = -\frac{gx^2}{2v^2}$,

$v = 25$ miles per hour $= 36\frac{2}{3}$ feet per second, $g = 32\frac{1}{6}$ feet per second;

$\therefore \quad 579x^2 = -48400y$, or $x^2 = -2ay$, suppose;

when $y = -5$, $x^2 = 10a$, or $x = (10a)^{\frac{1}{2}}$,

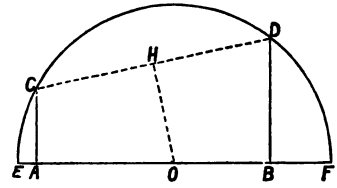
$$S = \frac{1}{a}\int_0^{(10a)^{\frac{1}{2}}} (a^2 + x^2)^{\frac{1}{2}}\,dx = \tfrac{1}{2}\left\{\sqrt{[10\,(a+10)]} + a\log_e \frac{\sqrt{(10)} + \sqrt{(a+10)}}{\sqrt{a}}\right\},$$

$S = 21{\cdot}233$ feet.

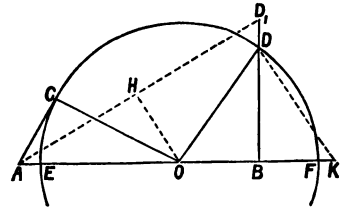
635. (Rev. T. GASKIN.)—(1) In the line AB to find two points E, F, such that $EA \times AF = AC^2$ and $BE \times BF = BD^2$, when AC and BD are given; E and F being also on different sides of A and B respectively. (2) To find E, F when E and F are on the same side of A, but on different sides of B. (3) To find E and F when they lie wholly between A and B.

Solution by K. S. PUTNAM.

(1) At A and B erect the perpendiculars AC and BD. From H, the centre of CD, erect a perpendicular intersecting AB. With O, the point of intersection, as a centre, describe the circle passing through C and D. Then E and F, the points of intersection with AB produced, will be the points required, for $AC^2 = AF \cdot AF$ and $BD^2 = BE \cdot BF$.



(2) From B erect perpendicular BD. Prolong AB, making $BK = AC$. Prolong BD to D_1, making $BD_1 = DK$. At H, the centre of AD_1, erect perpendicular, and, with O, its intersection with AB, as centre and OD as radius, describe a circle. E and F, its points of intersection with AB, will be the required points. $BD^2 = BE \cdot BF$, as in (1). If AC is tangent, it will be a mean proportional between the secant AF and external segment AE.



If AC is a tangent,

$$AO^2 = AC^2 + CO^2.$$

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But $OD^2 = OB^2 + BD^2$, and $CO = OD$. Then we would have

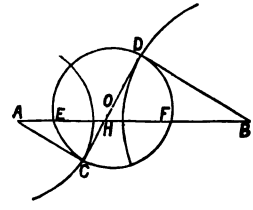
$$AO^2 = AC^2 + BD^2 + OB^2 \quad \text{or} \quad AO^2 - OB^2 = AC^2 + BD^2.$$

Now $$AC^2 + BD^2 = DK^2 = BD_1^2$$

and $$AO^2 - OB^2 = (AO + OB)(AO - OB) = AB\,(AO - OB).$$

Therefore BD_1 is a mean proportional between AB and a line $= AO - OB$, for which we have no further use, as we know that the centre of the circle passing through A and D_1, and having BD_1 as a semi-chord perpendicular to its diameter, AB produced, will be at the intersection of the perpendicular HO and such diameter. This point is O, which we have taken as centre of circle FDCE. AC is therefore perpendicular to CO and tangent, and $AC^2 = AE \,.\, AF$.

(3) If AB is the given line, and DC the common tangent to circle AC and circle BD, the circle whose diameter is DC will cut AB at E and F, the required points. For, AC being tangent to CFDE, $AC^2 = AF \,.\, AE$. Similarly $BD^2 = BE \,.\, BF$.



Construction.—Since the triangles ACH and BDH are right-angled, and have equal angles at H, they are equiangular and similar. AH : HB :: AC : BD. The point H is therefore formed by dividing AB into parts proportional to AC and BD. At one extremity of a line equal to BD erect an indefinite perpendicular, and from the other extremity, with a radius equal to BH, describe a circle cutting the perpendicular. This will give the distance HD; hence the point D. The point C is similarly found. From O, the centre of CD, the circle CEDF is described.

3854. (Professor Sir R. S. Ball, F.R.S.)—From any point perpendiculars are drawn to the generators of the surface $z(x^2 + y^2) - 2mxy = 0$. Show that the feet of the perpendiculars lie on a plane ellipse.

Solution by Professor Clayton, M.A.

Any generator is of form $x/\cos\theta = y/\sin\theta$, $z = m \sin 2\theta$.

Now, since locus point must satisfy the equations of a perpendicular or generator, we have these equations:

$$(x - x')\cos\theta + (y - y')\sin\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$x\sin\theta - y\cos\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

$$z - m\sin 2\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

From (1), (2), $$x^2 + y^2 - xx' - yy' = 0;$$

therefore one locus of point in question is this right circular cylinder.

Again, from (1), (3),

$$z(x^2 + y^2) - 2z(xx' + yy') + z(x'^2 + y'^2) + 2m(x - x')(y - y') = 0.$$

But, since also, for locus point, $z(x^2 + y^2) = 2mxy$,

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$\therefore \quad 4mxy - 2z(xx' + yy') + z(x'^2 + y'^2) - 2m(x'y + y'x - x'y') = 0;$

and, combining cylinder (above) with surface, we have

$$2mxy = z(xx' + yy');$$

hence, for locus of x, y, z,

$$z(x'^2 + y'^2) - 2m(xy' + x'y) + 2mx'y' = 0;$$

therefore complete locus is the intersection of this plane with above cylinder, *i.e.*, an ellipse.

Note.—Since the generators of surface cut the axis of z perpendicularly, they lie in planes passing through it. But the feet of perpendiculars from any point on these planes obviously lie on a circle, which is evidently the orthogonal projection of above ellipse.

[For another solution, see Vol. LXV., p. 28.]

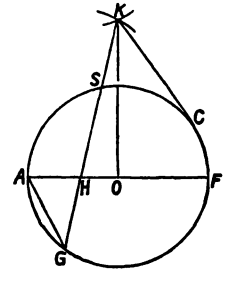
13332. (G. N. BATES, B.A.) — Prove the following method for inscribing a pentagon in a circle:—Let AFC be a circle, AF a diameter. Divide AF into five equal parts; with A and F as centres and radius AF, describe arcs of circles cutting in K. Join K with H, the second point of division; produce KH to cut circle in G. AG is the side of the pentagon.—Show that, in a circle of radius 1", the side first obtained is of length 1·174...". (True length should be 1·1755...".)

Solution by the PROPOSER.

Let R be radius of circle. Then

$$AH = \tfrac{1}{5}R, \quad KO = \sqrt{3}\,R,$$

$$KH = \frac{2\sqrt{(19)}}{5}R, \quad \cos KHO = \frac{\sqrt{(19)}}{38},$$



Now $KS \cdot KG = KC^2 = 2R^2$,

$SH \cdot HG = (24R^2)/25$,

and $KS \cdot KG = (KH - SH)(KH + HG) = 2R^2$;

$\therefore \quad SH - HG = -(2R^2 - \frac{16}{25}R^2 + \frac{24}{25}R^2)/HK$

$$= \frac{R}{5\sqrt{(19)}},$$

and $SH \cdot HG = \frac{24}{25}R^2$;

therefore $$HG = \frac{1}{10\sqrt{(19)}}\{5\sqrt{(73)} - 1\}R.$$

Now $$AG^2 = AH^2 + HG^2 - 2AH \cdot HG \cos AHG$$

$$= R^2\left\{\frac{16}{25} + \frac{1}{1900}[1826 - 10\sqrt{(73)}] - \frac{8}{50\sqrt{(19)}} \cdot \frac{5\sqrt{(73)} - 1}{2\sqrt{(19)}}\right\}$$

$$= \frac{R^2}{38}\{61 - \sqrt{(73)}\}.$$

Now, when R = 1", $AG^2 = 1{\cdot}380421$ app.;

therefore $AG = 1{\cdot}1749"$.

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5447. (Rev. W. Roberts, M.A.)—The cone $x^2\cot^2\alpha + y^2\cot^2\beta - z^2 = 0$ intersects the sphere $x^2+y^2+z^2-a^2=0$ in a sphero-conic. Show that the equation of the tubular surface, which is the envelope of a sphere of constant radius k, whose centre moves along this sphero-conic, is had by equating to zero the discriminant of the following cubic in λ,

$$4a^2\sin^2\alpha x^2/(P^2+4\lambda a^2\cos^2\alpha) + 4a^2\sin^2\beta y^2/(P^2+4\lambda a^2\cos^2\beta) - z^2/\lambda = 1,$$

where $$P = x^2+y^2+z^2+a^2-k^2.$$

Solution by Professor W. Booth.

Let $x'y'z'$ be the variable centre of the sphere; then

$$(x-x')^2+(y-y')^2+(z-z')^2 = k^2 \quad \ldots\ldots(1),$$

and $$x'^2+y'^2+z'^2 = a^2,$$

also $$z'^2 = x'^2\cot^2\alpha + y'^2\cot^2\beta.$$

Equation (1) is $x^2+y^2+z^2+a^2-k^2 = 2xx'+2yy'+2zz' = P$;

hence we obtain $$2x + 2\lambda' x' + 2\mu'\cot^2\alpha \,.\, x' = 0,$$

λ' and μ' being "undetermined multipliers," or

$$x = \lambda x' + \mu x'\cot^2\alpha, \quad y = \lambda y' + \mu y'\cot^2\beta, \quad z = \lambda x' - \mu z';$$

hence $$xx'+yy'+zz' = \lambda a^2; \quad \therefore \quad \lambda = P/2a^2, \text{ by (1)};$$

therefore $$x' = \frac{2a^2x\sin^2\alpha}{P\sin^2\alpha + \mu\cos^2\alpha \,.\, 2a^2},$$

with similar values for y' and z'. Substituting in

$$2xx'+2yy'+2zz' = P,$$

we have $$\frac{4a^2x^2\sin^2\alpha}{P^2+4\theta a^2\cos^2\alpha} + \frac{4a^2\sin^2\beta y^2}{P^2+4\theta a^2\cos^2\beta} - \frac{z^2}{\theta} = 1,$$

where θ is $\left(\frac{2\mu a^2 P - P^2}{4a^2}\right)$, and substituting the values of $x'y'z'$. Similarly in $z'^2 = x'^2\cot^2\alpha + y'^2\cot^2\beta$, we obtain in like manner

$$\frac{x^2\,.\,16a^4\sin^2\alpha\cos^2\alpha}{(P^2+4\theta a^2\cos^2\alpha)^2} + \frac{y^2\,.\,16a^4\sin^2\beta\cos^2\beta}{(P^2+4\theta a^2\cos^2\beta)^2} - \frac{z^2}{\theta^2} = 0.$$

Now this equation is manifestly the derived equation of the preceding with reference to θ; hence the discriminant of the former cubic in θ equated to zero is the result independent of $x'y'z'$, λ and μ, and is therefore the equation of the tubular surface required.

13392. (P. W. Flood.)—From a given point in the diameter produced of a given semicircle, draw a line cutting the circumference in two points such that, if perpendiculars be drawn from them on the diameter, their rectangle will be given.

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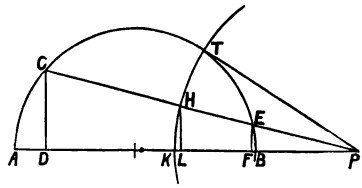
Solution by MORGAN BRIERLEY; W. E. JEFFARES, B.A.; *and others.*



Let ATB be the given semicircle, and P the given point in AB produced; from P draw the tangent PT, and, with PT as radius, describe the arc TK, cutting the diameter in K. On the diameter erect the perpendicular IH whose square equals the given rectangle, meeting this arc in H, through which draw PEHC, cutting the semicircle in EC; it will be the line required.

$$PK^2 = PT^2 = PA \cdot PB = PC \cdot PE = PH^2;$$

therefore $PC : PH = PH : PE = CD : HI = HI : EF,$

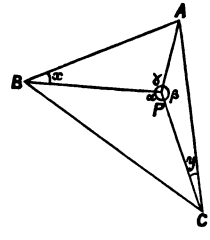
by similar triangles; therefore $CD \cdot EF = IH^2$.

1207. (N'IMPORTE.)—To determine the position of a rock (R), the angles subtended at it by the distances between three headlands (A, B, C) were observed, viz.,

$$BRC = 150° 2' 42'', \quad CRA = 133° 25' 57'', \quad ARB = 75° 31' 21'';$$

and it was known, from a previous survey, that AB = 8883, BC = 9870, and CA = 10857 yards. Find the distance of the rock from each of the headlands.

Solution by H. J. WOODALL, A.R.C.S.



This is POTHENOT'S problem (HOBSON, *Trigonometry*, page 177). For notation see figure.

Therefore $(B-x)+(C-y)+\alpha = \pi,$

but $B+C = \pi-A;$

therefore $x+y = \alpha-A.$

Again, $PA = c \sin x / \sin \gamma = b \sin y / \sin \beta;$

then put $\tan \phi = c \sin \beta / b \sin \gamma;$

therefore $\sin y / \sin x = \tan \phi;$

whence $(\sin x - \sin y)/(\sin x + \sin y)$

$= \tan (45° - \phi) = \tan \frac{1}{2}(x-y) / \tan \frac{1}{2}(x+y);$

$\therefore \tan \frac{1}{2}(x-y) = \tan \frac{1}{2}(x+y) \tan (45° - \phi) = \tan \frac{1}{2}(\alpha - A) \tan (45° - \phi).$

α, A, ϕ are known; we can find $\frac{1}{2}(x-y)$, and thence x and y. By actual work

$$A = 58° 59' 33'', \quad \tfrac{1}{2}(\alpha - A) = 46° 1' 34\tfrac{1}{2}'',$$

$$B = 70° 31' 44'', \quad C = 50° 28' 43'', \quad \phi = 31° 32' 5'', \quad 45 - \phi = 13° 27' 55'',$$

$$\tfrac{1}{2}(x-y) = 13° 56' 12\tfrac{1}{2}'', \quad x = 59° 57' 47'', \quad y = 32° 5' 22''.$$

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In $\triangle PCA$: $\angle PCA = 32° 5' 22''$, $\angle CPA = 133° 25' 57''$,
$\angle PAC = 14° 28' 41''$;
$\triangle PBA$: $\angle PBA = 59° 57' 47''$, $\angle BPA = 75° 31' 21''$,
$\angle PAB = 44° 30' 52''$;
$\triangle PBC$: $\angle PBC = 10° 33' 57''$, $\angle BPC = 151° 2' 42''$,
$\angle PCB = 18° 23' 21''$;
whence also $PA = 7942{\cdot}24$, $PB = 6432{\cdot}02$, $PC = 3737{\cdot}82$.

13337. (P. W. Flood.)—Euler says it would be difficult to find x so as to make $p^4 - 2p^2x^2 + x^4 + x$ a square; but show that it is not difficult.

Solution by J. W. West, Professor Mukhopadhyay, *and others.*

Writing the equation thus, $(p^2-x^2)^2 + x =$ a square, and eliminating (y) from the simultaneous equations

$$p^2-x^2 = y, \quad x = 2y + 1,$$

we obtain

$$p^2-x^2 = \tfrac{1}{2}(x-1);$$

$$\therefore \quad x = \tfrac{1}{2}\left\{\pm\left[(4p)^2 + 3^2\right]^{\frac{1}{2}} - 1\right\}.$$

13361. (W. C. Stanham.)—Construct (a) a right-angled triangle having its angular points on the given rectangular hyperbola $x^2 - y^2 = a^2$ (rectangular coordinates), and its centroid at the point $(\frac{1}{3}a\cos\phi, \frac{2}{3}a\tan\phi)$, ϕ being any given angle; and show (b) that, both in this case and in the case in which the centroid is to be at the point $(\frac{2}{3}a\sec\phi, \frac{1}{3}a\cot\phi)$, the determination of the coordinates of the angular points of one such triangle will not necessitate the solution of an equation of a degree higher than the second.

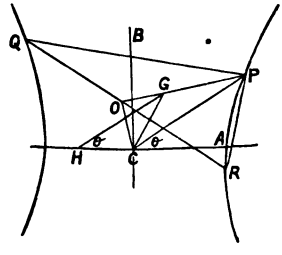
Solution by the Proposer, Professor Radhakrishnan, *and others.*

Let G be the point

$$(\tfrac{1}{3}a\cos\phi, \ \tfrac{2}{3}a\tan\phi),$$

or (x, y), and let

$$CG = r \quad \text{and} \quad \angle GCA = 2\theta.$$



(a) Draw CP to the curve, bisecting GCA; CO at right angles to CP, meeting PG in O; and QOR parallel to the diameter conjugate to CO. Then PQR is the triangle required. For

$$OR^2 = OQ^2 = CO^2 + CP^2 = OP^2,$$

since conjugate diameters (*sic*) are equal, and diameters at right angles

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(*sic*) are equal. Therefore PQR is a right-angled triangle. Also

$$9r^2 = a^2\cos^2\phi + 4a^2\tan^2\phi; \quad \therefore \quad a\cos\phi\,(3r + a\cos\phi) = 2a^2.$$

Draw GH parallel to CP; then $GH^2 = 2r\,(r+x)$;

$$\therefore \quad GH^2 = \tfrac{2}{3}a^2 \times 2r/x = \tfrac{4}{9}\sec 2\theta\, a^2 = \tfrac{4}{9}CP^2.$$

And CO bisects GH; $\quad \therefore \quad OP = 3OG$;
therefore G is the centroid of PQR.

(*b*) When G is the point $(\frac{2}{3}a\sec\phi, \frac{1}{3}a\cot\phi)$, CP must be drawn bisecting the obtuse angle between CG and CB. In both cases, P, Q, and R are determined by drawing straight lines to the curve. Therefore, &c.

13385. (H. B. Billups, B.A.)—Prove that (1) a, $-b$, $-c$ are the roots of the equation

$$x^3 + 2(s-a)x^2 + \left\{(s-a)^2 + r_a^2 - 4Rr_a\right\}x - 4Rr_a(-a) = 0;$$

and (2) $\cos A$, $-\cos B$, $-\cos C$ are the roots of

$$4R^2x^3 + 4R(r_a - R)x^2 + \left\{(s-a)^2 + r_a^2 - 4R^2\right\}x - (s-a)^2 + (2R - r_a)^2 = 0,$$

so that the triangle is acute-angled or obtuse-angled according as $s-a$ is numerically greater or less than $2R - r_a$.

13358. (H. B. Billups, B.A.)—P is the orthocentre of a triangle; show that PA, $-$PB, $-$PC are the roots of the equation

$$x^3 + 2(r_a - R)x^2 + \left\{r_a^2 - 4R^2 + (s-a)^2\right\}x + 2R\left\{(2R - r_a)^2 - (s-a)^2\right\} = 0.$$

Solution by the Proposer.

(13385) $y^3 - fy^2 + gy - h = 0$ has roots $-s$, $s-c$, $s-b$, where

$$f = -s + s - c + s - b = a - s,$$

$$g = -s(s-c) + (s-c)(s-b) - s(s-b) = \frac{\Delta^2}{rr_br_c}(-r_b + r - r_c) = r_a(r_a - 4R),$$

$$h = -s(s-c)(s-b) = -\frac{\Delta^2}{s-a} = -r_a^2(s-a);$$

$$\therefore \quad y^3 + (s-a)y^2 + r_a(r_a - 4R)y + r_a^2(s-a) = 0$$

has roots $\qquad -s,\ s-c,\ s-b.$

Write $y = a - s - x$. Then, when $y = -s$, $x = a$, &c.; therefore a, $-b$, $-c$ are the roots of

$$x^3 + 2(s-a)x^2 + \left\{(s-a)^2 + r_a^2 - 4Rr_a\right\}x - 4Rr_a(s-a) = 0.$$

$$\cos A - \cos B - \cos C = 1 - 2\sin\frac{A}{2}\left(\cos\frac{B+C}{2} + \cos\frac{B-C}{2}\right)$$

$$= 1 - 4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{R - r_a}{R},$$

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$$\Pi \cos A = \tfrac{1}{2} - \tfrac{1}{2}\Sigma \cos^2 A = -1 + \tfrac{1}{2}\Sigma \sin^2 A$$

$$= -1 + \frac{1}{8R^2}\Sigma a^2$$

$$= -1 + \frac{1}{8R^2}\left\{4(s-a)^2 - 2\left[(s-a)^2 + r_a^2 - 4Rr_a\right]\right\}$$

(by first part of question)

$$= \frac{(s-a)^2 - (r_a - 2R)^2}{4R^2}.$$

Again,

$$\cos B \cos C - \cos A \cos B - \cos A \cos C$$

$$= \tfrac{1}{2}(\cos A - \cos B - \cos C)^2 - \tfrac{1}{2}\Sigma \cos^2 A$$

$$= \tfrac{1}{2}\left(\frac{R - r_a}{R}\right)^2 + \Pi \cos A - \tfrac{1}{2}$$

$$= \tfrac{1}{2}\left(\frac{R - r_a}{R}\right)^2 - \tfrac{1}{2} + \frac{(s-a)^2 - (r_a - 2R)^2}{4R^2}$$

$$= \frac{(s-a)^2 + r_a^2 - 4R^2}{4R^2}.$$

If $s-a$ is numerically less than $2R - r_a$, one cosine is negative; and therefore the triangle is obtuse-angled.

(13358) $PA = 2R \cos A$, &c. The required results then follow at once from Quest. 13385.

[W. J. Greenstreet, M.A.; W. E. Jeffares, B.A.; *and others* solve Quest. 13358 as follows:—

$$r_a = 4R \cos \frac{B}{2} \cos \frac{C}{2} \sin \frac{A}{2} = R(1 + \cos B + \cos C - \cos A),$$

$$s - a = \tfrac{1}{2}(b + c - a) = R(\sin B + \sin C - \sin A)$$

$$-(PA - PB - PC)$$

$$= 2R(\cos B + \cos C - \cos A) = 2(r_a - R)$$

$$-PA.PB - PA.PC + PB.PC$$

$$= 4R^2(-\cos A \cos C - \cos A \cos B + \cos B \cos C)$$

$$= 2R^2\left[-\cos(A + C) - \cos(A - C), \text{ &c.}\right]$$

$$= R^2\left[2(\cos B + \cos C - \cos A) - 2(\cos A \cos B + \sin A \sin B) - \text{&c.}\right]$$

$$= R^2\left[(1 + \cos B + \cos C - \cos A)^2 + (\sin B + \sin C - \sin A)^2 - 4\right]$$

$$= R^2\left[r_a^2/R^2 - 4 + (s-a)^2/R^2\right] = r_a^2 - 4R^2 + (s-a)^2$$

$$-(PA)(-PB)(-PC)$$

$$= -8R^3 \Pi \cos A = +2R^3(1 + \Sigma \cos 2A)$$

$$= 2R^3\left[1 - 2(\cos B + \cos C - \cos A) + \Sigma(\cos^2 A - \sin^2 A)\right.$$

$$\left. - 2\left\{\cos(A + C) + \cos(A + B) - \cos(B + C)\right\}\right]$$

$$= 2R^3\left\{\left[1 - (\cos B + \cos C - \cos A)\right]^2 - (\sin B + \sin C - \sin A)^2\right\}$$

$$= 2R^3\left\{(-2 + r_a/R)^2 - (s-a)^2/R^2\right\} = 2R\left[(r_a - 2R)^2 - (s-a)^2\right];$$

therefore, &c.]

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Note on Quest. 5517. *By* H. W. CURJEL, M.A.

The Solution to this Question on page 25 of Vol. LX. is evidently wrong, OP being the semi-diameter down which a particle will fall in the same time as down OA.

If OR is the semi-diameter of quickest descent, R is evidently one of the points of contact of the circle (if there is one) passing through the centre and having double contact with the ellipse.

Hence at R the normal equals the distance of its point of intersection with the major axis from the centre.

Let ϕ = the excentric angle of R. Then

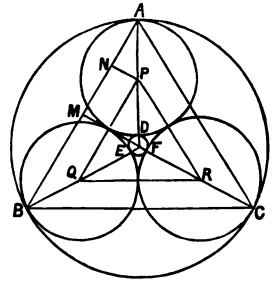
$$\left(\frac{a^2-b^2}{a}\cos\phi\right)^2 = \left(a\cos\phi - \frac{a^2-b^2}{a}\cos\phi\right)^2 + b^2\sin^2\phi;$$

therefore $$\tan\phi = \frac{\sqrt{(a^2-2b^2)}}{b}.$$

Hence, if $2b^2 > a^2$, *i.e.*, if $e < 1/\sqrt{2}$, the value of ϕ given by this equation is imaginary, and OA is evidently the semi-diameter of quickest descent. The point R may be constructed by producing OP (in the figure on page 25, Vol. LX.) to meet the auxiliary circle in r; then R is the point of intersection of the ellipse with a parallel to the minor axis through r.

13184. (P. W. FLOOD.)—In a circle 100 yards in diameter three equal circles are inscribed touching each other and the outer circle. It is required to find the diameter of a circle inscribed in the space at the centre.

Solution by Professors RADHAKRISHNAN, GOPALACHANAR, *and others.*



If R, r, ρ be the radii of the circles in order of magnitude, we have

$$AM = AN + NM = \tfrac{1}{2}r\,.\,\sqrt{3} + r.$$

Also $$R = SA = \tfrac{1}{2}AM\,.\,2\sqrt{3}$$

$$= \frac{r(2+\sqrt{3})}{\sqrt{3}};$$

$$\therefore\ r = \frac{R\,.\,\sqrt{3}}{2+\sqrt{3}};$$

$$\therefore\ SD = \rho = R - 2r$$

$$= R\left(1 - \frac{2\sqrt{3}}{2+\sqrt{3}}\right) = R\,\frac{2-\sqrt{3}}{2+\sqrt{3}}$$

$$= R(7-4\sqrt{3}) = R \times \cdot 0717968.$$

Therefore the diameter required = 7·18 yards.

[For another solution, see Vol. LXVI., pp. 80, 81.]

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3647. (J. B. SANDERS.)—A tube 30 inches long, and closed at one end and open at the other, was caused to descend in the sea with the open end downward, until the enclosed air occupied only one inch of the tube. How far did it descend?

Solution by H. J. WOODALL, A.R.C.S.

Assume BOYLE's law, also that density of sea at any depth = $a \times$ density of water at 4° C., pv = constant, $v = 30$, $p = 1$ atmosphere, $v = 1$,

$p = 30$ atmospheres = (1 atmo. + 29 × ht. of pure water barometer);

therefore tube will descend through 29 × ht. of pure water barometer $/a$.

13315. (Professor FINKEL.)—Find the average length of a line drawn at random across the opposite sides of a rectangle whose length is l, and breadth b.

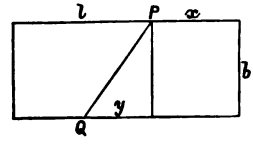
Solution by Rev. J. CULLEN, B.A.; Professor RAMASWAMI AIYAR; *and others.*

$$PQ = (b^2 + y^2)^{\frac{1}{2}}$$

$\therefore$ average length

$$= \int_0^b \int_0^{l-x} (b^2 + y^2)^{\frac{1}{2}} \, dx \, dy \Big/ \int_0^l \int_0^{l-x} dx \, dy$$

$$= (l^2 + b^2)^{\frac{1}{2}}/3l^2 - b^2 (l^2 + b^2)^{\frac{1}{2}}/l^2 + 2b^3/3l^2 + (b^2/l) \log \left[l + (l^2 + b^2)^{\frac{1}{2}} \right]/l.$$



Interchanging l and b the other case is obtained.

[See WILLIAMSON's *Integral Calculus*, Example 71, at end of volume.]

547. (STEPHEN WATSON.)—A uniform circular well, a feet deep, being filled with water, n men are successively employed to empty it: required the depth emptied by each, when an equal share of the work is done by each man.

Solution by K. S. PUTNAM.

In this problem we seem to be justified in disregarding the disposition to be made of the water and the question of inflow. For neither of these are there any data, and either would introduce elements varying widely. Considering the problem to contemplate simply the raising of a feet of water to the surface, the product of the quantity into the average height raised would represent, relatively, the work performed.

$a(\frac{1}{2}a) = \frac{1}{2}a^2$ = whole work. $a^2/2n$ = each man's share.

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Letting x, y, z = depth emptied by the first, second, and third men, respectively, we have

$$x(\tfrac{1}{2}x) = a^2/2n, \quad x = a/\sqrt{n},$$

$$y(\tfrac{1}{2}y + x) = a^2/2n, \quad y^2 + 2xy = a^2/n, \quad (y+x)^2 = 2a^2/n, \quad y = (a/\sqrt{n})(\sqrt{2}-1),$$

$$z(\tfrac{1}{2}z + x + y) = a^2/2n, \quad z^2 + 2(x+y)z = a^2/n, \quad (z+x+y)^2 = 3a^2/n,$$

$$z = (a/\sqrt{n})(\sqrt{3}-\sqrt{2}),$$

and generally $\quad m = (a/\sqrt{n})\{\sqrt{m} - \sqrt{(m-1)}\}.$

If $a = 16$ feet and $n = 4$, we have

1st empties	$\frac{16}{2}\sqrt{1}$	$= 8$	feet,
2nd ,,	$\frac{16}{2}(\sqrt{2}-\sqrt{1})$	$= 3{\cdot}3136$	,,
3rd ,,	$\frac{16}{2}(\sqrt{3}-\sqrt{2})$	$= 2{\cdot}5464$	,,
4th ,,	$\frac{16}{2}(\sqrt{4}-\sqrt{3})$	$= 2{\cdot}1400$	,,

13289. (D. Biddle.)—A sphere of plastic material is deformed by gradual flattening underneath, but so that the curved surface maintains sphericity of ever-increasing radius around a fixed centre, until a hemisphere results. Find the solid contents of the figure traced by the horizontal plane surface.

Solution by J. W. West; Rev. J. Cullen, B.A.; *and others.*

Let (r) be the radius of the primitive sphere; then

$$p^3 = \frac{4r^3}{(1+\cos\phi)^2(2-\cos\phi)}$$

is the polar equation to the directrix when the fixed centre is the pole; therefore

$$\text{contents} = \frac{8\pi r^3}{3}\int_0^{\frac{1}{2}\pi} \frac{d\phi \sin\phi}{(1+\cos\phi)^2(2-\cos\phi)}$$

$$= \frac{8\pi r^3}{27}\int_0^{\frac{1}{2}\pi} d\phi \sin\phi \left\{\frac{1}{(1+\cos\phi)} + \frac{1}{(2-\cos\phi)} + \frac{3}{(1+\cos\phi)^2}\right\}$$

$$= \frac{8\pi r^3}{27}\left\{\log_e\left(\frac{2-\cos\phi}{1+\cos\phi}\right) + \frac{3}{1+\cos\phi}\right\}_0^{\frac{1}{2}\pi} = \frac{4\pi r^3}{27}\{4\log_e 2 + 3\}$$

$$= 2{\cdot}68668r^3, \text{ nearly.}$$

13169. (Professor Orchard, M.A., B.Sc.)—Find the length of the evolute of the curve $p = r\sin\gamma$.

Solution by Rev. J. Cullen, B.A.; Prof. Ramaswami Aiyar; *and others.*

We have $\quad dr/dp = 1/\sin\gamma; \quad \therefore \quad \rho = r/\sin\gamma,$

and the length of the evolute $= \rho_1 - \rho_0 = (r_1 - r_0)/\sin\gamma$, or $r_1/\sin\gamma$ if measured from the origin. Both the curve and evolute are equiangular spirals.

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13335. (Prof. Beyens.)—Obtain a solution of $dy/dx + (ax + b)y = c$, where a, b, and c are constants; y to be expressed in terms of known functions of x, or as a rapidly convergent series of integral powers of x.

Solution by H. W. Curjel, M.A.; J. W. West; *and others.*

Multiplying by $e^{\frac{1}{2}a(x+b/a)^2}$, and integrating, we get

$$e^{\frac{1}{2}a(x+b/a)^2}y = d + c\int e^{\frac{1}{2}a(x+b/a)^2}dx = d + c\left\{x + \frac{b}{a} + \frac{a}{3.2}\left(x+\frac{b}{a}\right)^3 + \frac{a^2}{5(2!)2^2}\left(x+\frac{b}{a}\right)^5 + \ldots + \frac{1}{(2r+1)r!}\left(\frac{a}{2}\right)^r\left(x+\frac{b}{a}\right)^{2r+1} + \ldots\right\}.$$

13277. (Professor Cochez.)—Limite de $y = (x + \cos x + \tan x)^{1/(\sin x)}$ quand x tend vers zéro.

Solution by H. W. Curjel, M.A.; Professor Chakrivarti; *and others.*

Let the limit of y (when $x = 0$) $= u$. Then

$$\log u = \text{limit of } \frac{\log(x + \cos x + \tan x)}{\sin x}$$

$$= \text{limit of } \frac{1 - \sin x + \sec^2 x}{\cos x\,(x + \cos x + \tan x)} = 2;$$

$$\therefore \quad u = e^2.$$

13239. (J. F. Wilkinson.)—If x, y be the rectangular coordinates of a point and r its radius vector, prove that on each of a series of parabolas an indefinite number of points may be found giving integral values of x, y, r. Establish formulæ for values of x, y, r, in terms of an independent variable susceptible of integral values only.

Solution by Rev. J. L. Kitchin, M.A.; Prof. Sarkar; *and others.*

The coordinates of any point on a parabola can be represented by

$$x = am^2, \quad y = 2am, \quad \text{or by a single variable.}$$

Now $$r = a(1 + m^2).$$

So long as a is invariable we may get, by giving integral values to m, any number of points on a parabola for which x, y, r are integral.

If a varies also, we get the same result for a series of parabolas. Therefore &c., &c.

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13236. (H. W. Curjel, M.A.)—Prove that the diameters conjugate to a common diameter of two confocal conics are at right angles, and the sum of their squares is zero.

Solution by Rev. J. Cullen, B.A.; W. E. Heal, M.A.; *and others.*

Let the confocals be

$$x^2/a^2 + y^2/b^2 = 1, \quad x^2/(a^2-\lambda^2) + y^2/(b^2-\lambda^2) = 1 \ \ldots\ldots (1), (2);$$

$y = mx$, $y = m'x$, $y = m''x$ the common semi-diameter and its conjugates in (1) and (2); r, r', r'' their lengths. Then the points of intersection of (1) and (2) are given by

$$x^2 = a^2(a^2-\lambda^2)/(a^2-b^2) \quad \text{and} \quad y^2 = -b^2(b^2-\lambda^2)/(a^2-b^2);$$

also $mm' = -\dfrac{b^2}{a^2}$ and $mm'' = -\dfrac{b^2-\lambda^2}{a^2-\lambda^2}$ (Salmon's *Conics*, Art. 170);

$$\therefore \quad m^2m'm'' = \frac{b^2(b^2-\lambda^2)}{a^2(a^2-\lambda^2)};$$

but $$m^2 = \frac{y^2}{x^2} = -\frac{b^2(b^2-\lambda^2)}{a^2(a^2-\lambda^2)}, \quad \text{or} \quad 1 + m'm'' = 0;$$

hence the lines $y = m'x$ and $y = m''x$ are at right angles.

Again we have $r^2 + r'^2 = a^2 + b^2$, $r^2 + r''^2 = a^2 + b^2 - 2\lambda^2$;

$$\therefore \quad r'^2 + r''^2 = 2(a^2 + b^2 - \lambda^2 - r^2) = 2\left\{a^2 + b^2 - \lambda^2 - (x^2 + y^2)\right\} = 0.$$

13323. (Rev. T. Wiggins, B.A.)—Given, as an identity,

$$\int_0^{\frac{1}{2}\pi} (a + \sin^2 x)^n \cos x\, dx = \int_0^{\frac{1}{2}\pi} (1 + a - \sin^2 x)^n \sin x\, dx,$$ prove that

$$a^n + \frac{n}{3}a^{n-1} + \frac{n(n-1)}{2.5}a^{n-2} - \frac{n(n-1)(n-2)}{3!.7}a^{n-3} + \ldots$$

$$= (a+1)^n - \frac{2n}{3}(a+1)^{n-1} + \frac{2^2 n(n-1)}{3.5}(a+1)^{n-2}$$

$$- \frac{2^3 n(n-1)(n-2)}{3.5.7}(a+1)^{n-3} + \ldots.$$

Solution by Rev. J. Cullen, B.A.; H. W. Curjel, M.A.; *and others.*

Expanding, we have

$$\Sigma \frac{n!}{r!\,n!-r} a^{n-r} \int_0^{\frac{1}{2}\pi} \sin^{2r} x \cos x\, dx = \Sigma (-)^r \frac{n!}{r!\,n!-r} (a+1)^{n-r} \int_0^{\frac{1}{2}\pi} \sin^{2r+1} x\, dx.$$

Whence the stated result, since

$$\int_0^{\frac{1}{2}\pi} \sin^{2r} x \cos x\, dx = 1/(2r+1)$$

and $$\int_0^{\frac{1}{2}\pi} \sin^{2r+1} x\, dx = 2.4.6\ldots 2r/3.5.7\ldots(2r+1).$$

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3728. (Rev. A. F. Torry, M.A.)—Taking the eccentricity of the earth's orbit round the sun to be $\frac{1}{60}$, and the year to consist of $365\frac{1}{4}$ days, determine the greatest and least angular velocities of the earth about the sun.

Solution by Professor Ramachandra Row, M.A.

$$\omega = \frac{d\theta}{dt} = \frac{1}{r^2} r^2 \frac{d\theta}{dt} = \frac{h}{r^2} = \frac{\left[\mu a (1-e^2)\right]^{\frac{1}{2}}}{r^2}.$$

Again $\quad T = 2\pi \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}; \quad \therefore \quad \omega = \frac{2\pi}{T} (1-e^2)^{\frac{1}{2}} \left(\frac{a}{r}\right)^2.$

At the greatest angular velocity $r = a(1-e)$; least when $r = a(1+e)$;

$$\therefore \quad \omega_1 = \frac{2\pi}{T}\left[\frac{1+e}{(1-e)^3}\right]^{\frac{1}{2}} \quad \text{and} \quad \omega_2 = \frac{2\pi}{T}\left[\frac{1-e}{(1+e)^3}\right]^{\frac{1}{2}},$$ and are known, since T and e are given.

3223. (By W. Siverly.)—Find the greatest cube which can be cut from the solid generated by the revolution of a cycloid around its base.

Solution by H. J. Woodall, A.R.C.S.

$$x = a(\theta + \sin\theta), \quad r = (y^2 + z^2)^{\frac{1}{2}} = a(1 + \cos\theta);$$

whence we must solve

$$\sqrt{2}(\theta + \sin\theta) = 1 + \cos\theta, \quad \text{or} \quad \sin(\theta - \alpha) = \tfrac{1}{3}\left\{\sqrt{3} - \theta\sqrt{6}\right\},$$

where $\qquad \alpha = \sin^{-1}\frac{1}{3}\sqrt{3} = 35° 15' 53''.$

We find $\theta = 37° 37' 30''$ nearly.

12390. (S. Tebay, B.A.)—Find two rational fractions such that their sum shall be equal to the sum of their squares, which is also a square.

Solution by Morgan Brierley.

$x + y = x^2 + y^2$. Put $y = rx$; then

$$x + rx = x^2 + r^2x^2, \quad \text{or} \quad (r+1)x = (r^2+1)x^2;$$

$$\therefore \quad x = \frac{r+1}{r^2+1}, \quad y = \frac{r^2+r}{r^2+1},$$

$$x + y = \frac{(r+1)^2}{r^2+1}, \quad x^2 + y^2 = \frac{(r+1)^2}{r^2+1}.$$

Assume $r = \frac{7}{4}$; then

$$x + y = \frac{28}{25} + \frac{21}{25} = \frac{1225}{625} = \frac{(35)^2}{(25)^2} = x^2 + y^2.$$

[For another solution of this Question, see Vol. lxii., p. 70.]

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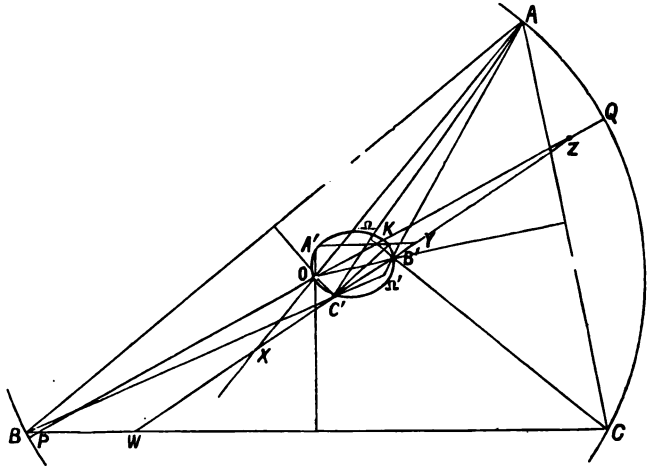
13311. (Professor SCHWATT.)—The SIMSON-line belonging to one point of intersection of BROCARD's diameter of a triangle with the circumcircle of this triangle is either parallel or perpendicular to the bisector of the angle formed by the side BC of the triangle ABC and the corresponding side B′C′ of BROCARD's triangle.

Solution by Professor SANJÁNA.



Lemma.—The angle made with any side of a triangle by the SIMSON-line of a point with regard to the triangle is half of the angle between the diameters drawn through the point and through the opposite vertex of the triangle. This may be proved by Euclid or elementary analytical geometry. Thus, in the accompanying figure, the SIMSON-line of P makes with BC an angle $= \frac{1}{2}AOQ$; hence, to prove it parallel to the bisector of the angle between BC and B′C′, we have to show that $C'WC = AOQ$. This can be done by trilinears; the equations of the containing sides being well known. The process, though straightforward, is tedious. The following is a purely geometrical proof:—

Let the seven-point circle (BROCARD's) go through A′, B′, C′, the vertices of BROCARD's triangle; through Ω, Ω′, BROCARD's points; and through the diametrical points O and K, the circumcentre and symmedian. Let AO, A′K, OK meet B′C′ in X, Y, Z. Then KA′, KB′, KC′ are parallel, and A′O, B′O, C′O perpendicular, to BC, CA, AB; hence $C'WC = A'YC'$, and $A'OB' = C$. Also $\Omega OK = \omega$, and $\Omega AO = \omega + C - \frac{1}{2}\pi$. Therefore

$$\begin{aligned} AOK &= AXC' + YZK = AC'B' - \Omega AO + A'YC' - YKZ \\ &= \Omega C'B' - \omega - C + \tfrac{1}{2}\pi + C'WC - A'KO \\ &= \Omega OB' - \Omega OK - A'OB' + C'WC + A'OK \\ &= (\Omega OB' - \Omega OK) - (A'OB' - A'OK) + C'WC = C'WC. \end{aligned}$$

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But the SIMSON-line of P makes with BC an angle $= \frac{1}{2}AOK$; and the bisector of the angle between BC and B′C′ makes with BC an angle $= \frac{1}{2}C'WC$; thus these two are parallel. If Q be the other extremity of the diameter of BROCARD's circle, the angle which its SIMSON-line makes with BC $= \frac{1}{2}AOP$; this line, therefore, is perpendicular to the bisector above.

[For another solution of this Question see Vol. LXVII., p. 32.]

13359. (C. E. HILLYER.)—If the polar of a point O with respect to a conic meet any pair of conjugate diameters in F and f, and EOe be drawn antiparallel to Ff with respect to the angle FCf meeting CF in E, and CF in e, prove that F, E and f, e are pairs of inverse points with respect to the director circle; hence show (1) that the circumcircle of a self-conjugate triangle is orthogonal to the director circle; (2) by taking O on the curve, that tangents at right angles intersect on the director circle. Examine the special cases of the parabola and rectangular hyperbola.

Solution by the PROPOSER.

Let PP′, DD′ be the conjugate diameters. Draw OK, Ok parallel to CD, CP, respectively. Then

$$\frac{CF}{Cf} = \frac{Ce}{CE} = \frac{OK}{EK};$$

therefore $CF \cdot EK = Cf \cdot OK = Cf \cdot Ck = CD^2$;

$\therefore$ $CF \cdot CE$
$= CF \cdot EK + CF \cdot CK$
$= CD^2 + CP^2$
$= CA^2 + CB^2 = R^2$,

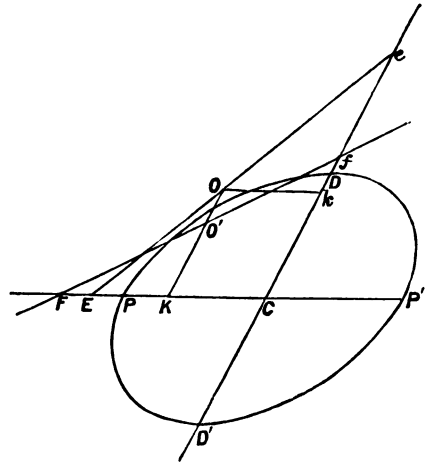
where R is the radius of the director circle; therefore F, E are inverse points with respect to the director circle, and similarly for f, e.

(1) If OK meets Ff in O′, the triangle OO′F is self-conjugate with respect to the conic; but, since

$\angle O'FE = Oef = EOO'$,

the circle OO′F passes through E, and is orthogonal to the director circle, since

$CE \cdot CF = R^2$.



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(2) When O is on the curve Ff becomes the tangent at O. Now E, F will coincide, and therefore both lie on the director circle, when the tangent at O is equally inclined to CP and CD, but in this case the tangents from F to the conic will be at right angles, being parallel to the bisectors of $\angle$ PCD; therefore tangents at right angles intersect on the director circle.

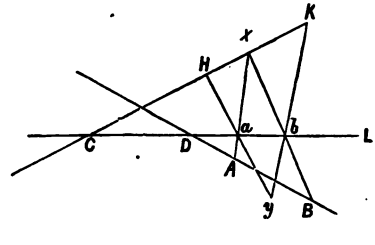
In the case of the parabola the directrix bisects EF, C being at an infinite distance but the *direction* of CD remaining that of the tangent at P.

In the case of the rectangular hyperbola CE and Ce vanish since R = 0, that is, EOe coincides with OC.

13368. (W. J. Dobbs, M.A.)—A, B, H, K are four fixed points, and L a fixed straight line. Any point X is taken in HK, and AX, BX meet the line L in a, b respectively; Ha, Kb intersect in Y. Prove that, as X traces out the line HK, Y traces out a straight line which passes through the intersection of AB, ab.

Solution by Professor A. Droz-Farny, W. C. Stanham, *and others.*

Lorsque X se meut sur HK, AX et BX décrivent deux faisceaux perspectifs; a et b décrivent donc sur L deux ponctuelles homographiques ayant un point double en D, point d'intersection de L et AB. Il en résulte que les faisceaux Ha et Kb sont homographiques et que le lieu de Y est une courbe passant par D. Lorsque X coincide avec C, point d'intersection de HK et L, les deux rayons homologues Ha et Kb coincident avec la ligne des centres HK; les deux faisceaux Ha et Kb sont donc perspectifs, et le lieu de y est une ligne droite passant par D.



13297. (Rev. T. Wood Robinson, R.N.)—Prove that a rhumb line may be projected into an equiangular spiral; find its equation and deduce the formula tan course = $\frac{\text{diff. long.}}{\text{mer. diff. lat.}}$.

Solution by the Proposer.

If we project stereographically from one of the poles, the meridians become equally inclined straight lines meeting in a point—other pole—

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and, since angles are unaltered by this projection, the projection of the rhumb line must make equal angles with these straight lines, and will therefore be an equiangular spiral. Let (L, l) and L′l' be longitude and latitude of two places joined by rhumb line, and let $r = ce^{\theta \cot \alpha}$ be equation to spiral, where α will be the course on rhumb line; then, from progression, we have

$$\text{D} \tan \tfrac{1}{2}(\tfrac{1}{2}\pi + l) = ce^{\text{L} \cot \alpha}, \quad \text{and} \quad \text{D} \tan \tfrac{1}{2}(\tfrac{1}{2}\pi + l') = ce^{\text{L}' \cot \alpha};$$

therefore
$$\frac{\tan \tfrac{1}{2}(\tfrac{1}{2}\pi + l)}{\tan \tfrac{1}{2}(\tfrac{1}{2}\pi + l')} = e^{(\text{L} - \text{L}') \cot \alpha}.$$

Taking logs,
$$\tan \alpha = \frac{\text{L} - \text{L}'}{\log(\tfrac{1}{2}\pi + \tfrac{1}{2}l) - \log(\tfrac{1}{2}\pi + \tfrac{1}{2}l')};$$

therefore
$$\tan \text{course} = \frac{\text{diff. long.}}{\text{mer. diff. lat.}}.$$

[See CHAMBERS' *Practical Mathematics*, Edit. 1894, p. 470, art. 927.]

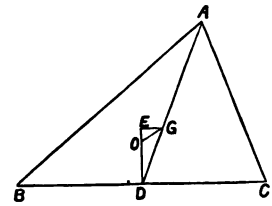
13371. (R. W. D. CHRISTIE.)—If G, O, a, b, c are the centroid, circumcentre, and sides of any triangle, prove that

(1) $36\text{GO}^2 = 9a^2 \cot^2 \text{A} + 5a^2 - 4b^2 - 4c^2$;

and (2), if the triangle be right-angled, GO = R/3.

Solution by W. E. JEFFARES, B.A.; H. W. CURJEL, M.A.; *and others.*

Let D be the middle point of BC, and E the foot of the perpendicular from G on DO. Then



$$36\text{GO}^2 = 36\text{OD}^2 + 36\text{GD}^2 - 72\text{DO} \cdot \text{DE}$$
$$= 9a^2 \cot^2 \text{A} + (2b^2 + 2c^2 - a^2) - 72 \cdot \frac{a \cos \text{A}}{2 \sin \text{A}} \cdot \tfrac{1}{2} b \sin \text{C}$$
$$= 9a^2 \cot^2 \text{A} + 2b^2 + 2c^2 - a^2 - 12bc \cos \text{A}$$
$$= 9a^2 \cot^2 \text{A} + 2b^2 + 2c^2 - a^2 - 6(b^2 + c^2 - a^2)$$
$$= 9a^2 \cot^2 \text{A} + 5a^2 - 4b^2 - 4c^2;$$

$\therefore$ when A is a right angle, $36\text{GO}^2 = 5a^2 - 4a^2 = 4\text{R}^2$;

$\therefore$ GO = R/3.

13322. (Rev. J. CULLEN, B.A.)—Prove that

$$\int_0^{\text{K}} \text{cn}^2 u (\text{cn}^2 u \, \text{dn}^4 u + 2k'^2 \text{sn}^2 u) \, du \Big/ \text{dn}^4 u = \left\{2(1 + k^2)\text{E} - 2\text{K} - k^2\text{K} + 3k^4\text{K}\right\} \Big/ 3k^4.$$

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Solution by H. W. CURJEL, M.A.; Professor CHAKRIVARTI; *and others.*

Write s, c, d for $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$.

Then $k^2 \frac{d}{du}(scd^m) = k^2(c^2 - s^2)\, d^{m+1} - mk^4 s^2 c^2 d^{m-1}$

$= (2d^2 - k'^2 - 1)\, d^{m+1} - md^{m-1}\{-d^4 + (1+k'^2)\, d^2 - k'\}$

$= (2+m)\, d^{m+3} - (m+1)(1+k'^2)\, d^{m+1} + mk'^2 d^{m-1}$,

and $scd^m = 0$, when $u = 0$ and when $u = \mathrm{K}$;

$\therefore \int_0^{\mathrm{K}} \frac{3k^4 s^2 c^2}{d^4}\, du = \int_0^{\mathrm{K}} \left(\frac{k'^2+1}{d^2} - 2\right) du$, and $\int_0^{\mathrm{K}} \frac{k'^2}{d^2}\, du = \int_0^{\mathrm{K}} d^2 du = \mathrm{E}$,

and $\int_0^{\mathrm{K}} 3d^4 du = \int_0^{\mathrm{K}} \{2(1+k'^2)\, d^2 - k'^2\}\, du$;

therefore $3k^4 \times$ given integral

$= \int_0^{\mathrm{K}} \left\{3d^4 - 6d^2 k'^2 + 3k'^4 + 2k'^2\left(\frac{1+k'^2}{d^2} - 2\right)\right\} du$

$= \int_0^{\mathrm{K}} \{2(2-k'^2)\, d^2 + k'^2(3k'^2 - 5)\}\, du$

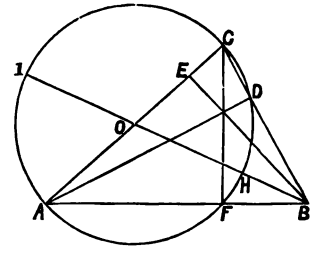
$= 2(1+k^2)\,\mathrm{E} - (1-k^2)(2+3k^2)\,\mathrm{K}$;

therefore given integral

$= \{2(1+k^2)\,\mathrm{E} - 2\mathrm{K} - k^2\mathrm{K} + 3k^4\mathrm{K}\}/3k^4$.

13390. (D. BIDDLE.)—Construct the triangle of which AD, BE, CF are the unknown perpendiculars, having given the three products BC.CD, CA.AE, AB.BF.

Solution by MORGAN BRIERLEY.



A, E, D, B are points in the arc of a circle; therefore

$\mathrm{AC}.\mathrm{CE} = \mathrm{BC}.\mathrm{CD} = a$,

and $\mathrm{AC}.\mathrm{AE} = b$;

hence $\mathrm{AC}(\mathrm{AE} + \mathrm{CE}) = a + b = \mathrm{AC}^2$;

therefore AC is given. Again,

$\mathrm{AC}(\mathrm{AE} - \mathrm{CE}) = a - b$;

$\therefore \mathrm{AE} - \mathrm{CE} = (a-b)/\mathrm{AC}$,

and AE, EC are given. With centre O, the middle of AC, describe a circle which will pass through D and F, and draw BHOI intersecting it in H and I; then $\mathrm{BH}.\mathrm{BI} = \mathrm{BD}.\mathrm{BC} = \mathrm{BF}.\mathrm{BA} = c$; III = AC, being known, BH becomes known, and, since $\mathrm{AB}^2 - \mathrm{BC}^2 = \mathrm{AE}^2 - \mathrm{EC}^2$, the point B is found, and the construction manifest.

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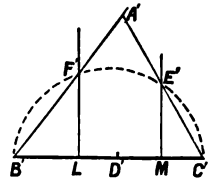
Solution by LIONEL E. REAY.

We are given BC.CD; that is, CD.CB, which = CE.CA, and we know CA.AE; therefore we know ratio AE : EC, and similarly CD : DB, BF : FD. Draw any line B′C′; divide at D′ in ratio BD : DC. Divide B′D′ in ratio BF : FA, D′C′ in ratio AE : EC. Describe a semicircle on B′C′. Perpendiculars through L and M meet it at F′ and E′. B′F′ and C′E′ meet at A′. A′B′C′ is similar to triangle required, and is such that squares on sides of A′B′C′ : squares on sides of ABC :: B′C′.C′D′ : BC.CD; therefore, &c.



[Mr. CULLEN solves the Question as follows:—

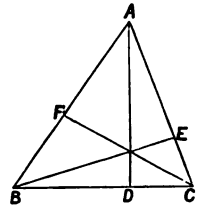
$$2BC.CD = BC^2 + AC^2 - AB^2,$$

$$2AC.AE = AC^2 + AB^2 - BC^2;$$

therefore $BC.CD + AC.AE = AC^2$

$= a$ given quantity,

and similarly for AB and BC. Hence the construction is obvious.]



4440. (The late Professor WOLSTENHOLME, M.A., Sc.D.)—If $x_1, x_2, \ldots x_n$ be connected by the equation

$$x_1^2 + x_2^2 + \ldots + x_n^2 - x_1x_2 - x_2x_3 + \ldots - x_{n-1}x_n = 1,$$

show that the maximum and minimum values of $x_1^2 + x_2^2 + \ldots x_n^2$ will be the values of $\left(1 + \cos\frac{r\pi}{n+1}\right)^{-1}$, where r has all integral values from 1 to n.

Solution by Rev. J. CULLEN, B.A.

Let $u = \Sigma x_1^2$, then $du = 2\Sigma x_1 dx_1 = 0$, or

$$\Sigma x_1 dx_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

also $$2\Sigma x_1 dx_1 - d\Sigma x_1 x_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence, employing the usual method undetermined multipliers, we get $u = 2/(\mu + 2)$, where μ is a root of the continuant,

$$\begin{vmatrix} \mu, & 1, & 0, & 0, & 0, & \ldots\ldots\ldots & \\ 1, & \mu, & 1, & 0, & 0, & \ldots\ldots\ldots & \\ 0, & 1, & \mu, & 1, & 0, & \ldots\ldots\ldots & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ 0, & 0, & \ldots\ldots & 0, & 1, & \mu, & 1 \\ 0, & 0, & \ldots\ldots & 0, & 0, & 1, & \mu \end{vmatrix} = 0,$$

But, by Quest. 7574, Vol. XLI., μ is of the form $2\cos\frac{r\pi}{n+1}$; therefore $u = \left(1 + \cos\frac{r\pi}{n+1}\right)^{-1}$ as stated.

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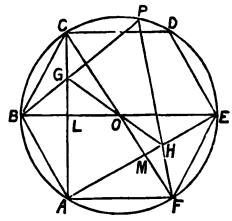
13356. (Professor PERCÉPIED.)—Par le centre d'un hexagone régulier ABCDEF, on mène une sécante quelconque qui coupe en G, H les diagonales AC, AE. Trouver le lieu du point d'intersection des droites BG et FH.

Solution by Professor ANDERSON; W. J. GREENSTREET, M.A.; *and others.*



B and F are the images of O in the sides AC and AE. Hence, if P be the point of intersection, the path of a ray of light from P through O after reflection at both mirrors will be PHOGP. But the deviation is twice the angle between the mirrors. Hence the angle BPF is 60°, and the locus is the circumscribing circle of the hexagon.

[Professor A. DROZ-FARNY sends an alternative solution as follows:—Considérons l'hexagone ACFPBE. AC et PB se coupent en G, CF et BE se coupent en O, FP et AE se coupent en H. G, O, H étant en ligne droite, l'hexagone est inscrit dans une conique bien déterminée par les cinq points A, C, F, B, E. Le lieu de P est donc bien la circonférence circonscrite à l'hexagone régulier.]

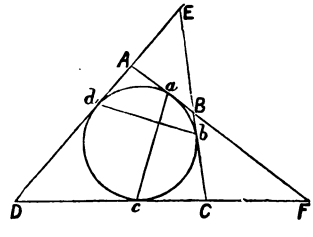
13401. (Professor NEUBERG.)—Les côtés AB, BC, CD, DA d'un quadrilatère ABCD touchent une circonférence aux points a, b, c, d.

Démontrer que $$\frac{(ac)^2}{(bd)^2} = \frac{AD.BC}{AB.DC}.$$

Solution by G. HEPPEL, M.A.; Professor A. DROZ-FARNY; *and others.*

Let the sides in order be h, k, l, m. Then, in the figure, since the circle is escribed to EAB,

$$bd = 2h \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B,$$



where A, B are angles of the quadrilateral. Similarly

$$ac = 2k \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C;$$

$$\therefore \quad \sin \tfrac{1}{2}A / \sin \tfrac{1}{2}C = k \,.\, bd/h \,.\, ac.$$

Now

$$BD^2 = (h-m)^2 + 4hm \sin^2 \tfrac{1}{2}A$$
$$= (k-l)^2 + 4kl \sin^2 \tfrac{1}{2}C.$$

Also $$h - m = k - l; \quad \therefore \quad \sin^2 \tfrac{1}{2}A / \sin^2 \tfrac{1}{2}C = kl/hm,$$

whence $$(ac)^2/(bd)^2 = km/hl.$$

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13192. (Professor MORLEY.)—A. can dig a field of potatoes in one hour, and pick the roots from the plants in two hours. B. can dig that field in three hours and pick it in four. C. can dig in five hours and pick in six. What is the least time in which all working together can dig and pick the field? Is a general formula known or obtainable for the least time in which m men can do n things, where no two things can be profitably done simultaneously by the same man?

Solution by Rev. T. ROACH, M.A.; Prof. KRISHNAMACHARRY; *and others.*

A. has a greater advantage over the others in digging than in picking; therefore, as they cannot together pick faster than he digs, let him dig the whole.

Then, if x be the whole time, we have

$$\tfrac{1}{2}(x-1)+\tfrac{1}{4}x+\tfrac{1}{6}x=1\ ; \qquad \therefore\ x=1\tfrac{7}{11}\text{ hours.}$$

[Let a_1, b_1, x_1, y_1 be respectively the time in which A. can dig the whole field, the time in which he can pick the whole, the actual proportion of it he digs, the actual proportion of it he picks; and let a_2, b_2, x_2, y_2 be the similar quantities of B., and so on. Then we have

$$x_1+x_2+x_3=1, \quad y_1+y_2+y_3=1,$$

and $$a_1x_1+b_1y_1=a_2x_2+b_2y_2=a_3x_3+b_3y_3$$
$$=\text{the time occupied by the entire process,}$$

which must be a minimum. In the case before us, there is the peculiarity that the digging must proceed in advance of the picking, and there will be waste of time if the pickers overtake the diggers, one or more. Now, if n men be engaged at the same time on the same kind of work, and the time in which each could do the whole be known, then the time in which they can do it combinedly is given by the product of the n times, divided by the sum of products of the same taken $(n-1)$ at a time. In the present case, B. and C. together can pick the whole field in $b_2b_3/(b_2+b_3)$ hours, and, as this is $>a_1$, A. will finish digging the field by the time they have done $a_1(b_2+b_3)/b_2b_3$ of it in the way of picking. All three will pick the remainder in $\dfrac{b_1b_2b_3-a_1b_1(b_2+b_3)}{b_1b_2+b_1b_3+b_2b_3}$, or $\frac{7}{11}$ of an hour, agreeing with the above solution, when the one hour of A.'s digging is added. But it cannot hereby be certified which is the speediest method without trial of all:—

(1) ..... A., B., C. combining to dig the whole and pick the whole,
(2) ..... A., B. digging whilst C. picks, and then all picking,
(3) ..... A., C. ,, B. ,, ,, ,,
(4) ..... B., C. ,, A. ,, ,, ,,
(5) ..... A. ,, B., C. ,, ,, ,,
(6) ..... B. ,, A., C. ,, ,, ,,
(7) ..... C. ,, A., B. ,, ,, ,,

Now (1) yields 1·7431 hours, (2) $1\cdot70\dot{4}\dot{5}$, (3) $1\cdot6\dot{9}$, (4) 1·9432, (5) $1\ 6\dot{3}$, (6) ·3, because A., C. pick and have to wait for B to supply

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them, (7) 5, for a similar reason. Therefore (5) is the quickest method, and the above solution is confirmed. The minimum time, however, could have been found from the above five equations, for we get $11t = 22-4x_1-x_2 = 21-3x_1+x_3$, least when $x_1 = 1$, $x_2 = x_3 = 0$.]

4701. (W. S. B. Woolhouse, F.R.A.S.)—If, within a given closed area, three points be taken at random as the apices of a triangle, show that (1) the average of the square of the area of all such triangles will be reduced to one-third the value, if one of the points be fixed at the centre; also (2), if within a given volume of space four points be taken at random as the apices of a tetrahedron, the average of the square of the volume of all such tetrahedra will be reduced to one-fourth the value, if one point be fixed at the centre; and (3) that this theorem is true when the enclosed area or volume of space is *of any form whatever*.

Solution by H. W. Curjel, M.A.

Take the centre of gravity O as origin, and the principal axes at O as axes of coordinates.

Then (1) $\iint x\,dx\,dy = 0, \qquad \iint y\,dx\,dy = 0, \qquad \iint xy\,dx\,dy = 0,$

$$\iint x^2\,dx\,dy = Ah_1^2, \quad \text{and} \quad \iint y^2\,dx\,dy = Ah_2^2;$$

(2) $$\iiint x\,dx\,dy\,dz = 0, \ \&c., \qquad \iiint yz\,dx\,dy\,dz = 0, \ \&c.,$$

$$\iiint x^2\,dx\,dy\,dz = Vk_1^2, \quad \iiint y^2\,dx\,dy\,dz = Vk_2^2, \quad \iiint z^2\,dx\,dy\,dz = Vk_3^2,$$

where A and V are the area and volume of the figures, and h_1, h_2, k_1, k_2, k_3 the principal radii of gyration.

(1) Let the coordinates of P_n be x_n, y_n,

$$(\text{area of } \Delta P_1P_2P_3)^2 = \tfrac{1}{4}(x_1y_2-y_3x_1+\ldots\text{to 6 terms})^2,$$

a function of the second degree in x_ny_n $(n = 1, 2, 3)$.

Hence terms which are perfect squares are the only ones which do not vanish after the operation:

$$\iiiiiint (\ )\,dx_1dy_1dx_2dy_2dx_3dy_3,$$

for there is no term independent of the coordinates, and the term, $x_m^2y_n^2$, gives $A^3h_1^2h_2^2$; $\therefore$ mean square area of $P_1P_2P_3 = \dfrac{6A^3h_1^2h_2^2}{4A^3} = \dfrac{3h_1^2h_2^2}{2}$;

$$(\text{area of } \Delta OP_1P_2)^2 = \tfrac{1}{4}(x_1y_2-x_2y_1)^2 \quad (\text{two terms});$$

$\therefore$ mean square of area of $OP_1P_2 = \dfrac{2A^3h_1^2h_2^2}{4A^3} = \dfrac{h_1^2h_2^2}{2}$

$= \frac{1}{3} \times$ mean square of area of $P_1P_2P_3$.

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(2) (Vol. of tetrahedron $P_1P_2P_3P_4$)

$$= \tfrac{1}{36}\left\{x_1y_2z_3 - x_1y_3z_2 + \ldots \text{to 24 terms}\right\}^2;$$

$\therefore$ mean square of vol. of $P_1P_2P_3P_4 = \dfrac{24V^4k_1^2k_2^2k_3^2}{36V^4} = \tfrac{2}{3}k_1^2k_2^2k_3^2$.

$$(\text{Vol. of } OP_1P_2P_3)^2 = \tfrac{1}{36}\left\{x_1y_2z_3 - x_1y_3z_2 + \ldots \text{to 6 terms}\right\}^2;$$

$\therefore$ mean square of vol. $OP_1P_2P_3 = \dfrac{6k_1^2k_2^2k_3^2}{36} = \tfrac{1}{6}k_1^2k_2^2k_3^2$

$= \tfrac{1}{4}$ of mean square of vol. of $P_1P_2P_3P_4$.

Generally, the mean square of the hyper-vol. determined by $(n+1)$ random points in a figure of n dimensions $= (n+1)$ mean square when one point is fixed at the centre of gravity.

The figures clearly may be of any form whatever.

13198. (Professor SANJÁNA.)—Prove that the average area of a triangle on a given base (b), and having a given vertical angle (α), is $\tfrac{1}{4}b^2\left(\dfrac{1}{\tan\alpha} + \dfrac{1}{\pi-\alpha}\right)$. Prove also that the average area of a triangle inscribed in a circle of given radius (a) with its base in a fixed direction is $\dfrac{a^2}{\pi}\int_0^\pi \dfrac{d\theta}{\theta}\sin^2\theta$. Will this expression give the average area of any triangle inscribed in the circle?

Solution by H. W. CURJEL, M.A.; Prof. MUKHOPADHYAY; *and others.*

Let radius of circumcircle $= a$. Then $a = \tfrac{1}{2}b\operatorname{cosec}\alpha$.

$$\text{Average area} = \left[\int_0^{\frac{1}{2}(\pi-\alpha)} \tfrac{1}{2}ba\left\{\cos\alpha - \cos(2\theta-\alpha)\right\}d\theta\right] \Big/ \left\{\tfrac{1}{2}(\pi-\alpha)\right\}$$

$$= \frac{ba}{\pi-\alpha}\left\{\tfrac{1}{2}(\pi-\alpha)\cos\alpha + \tfrac{1}{2}\sin\alpha\right\}$$

$$= \tfrac{1}{4}b^2\left(\frac{1}{\tan\alpha} + \frac{1}{\pi-\alpha}\right) = \left(\sin\alpha\cos\alpha + \frac{\sin^2\alpha}{\pi-\alpha}\right)a^2.$$

Hence, in second case, the average area

$$= \frac{a^2}{\pi}\int_0^\pi \left(\sin\theta\cos\theta + \frac{\sin^2\theta}{\pi-\theta}\right)d\theta$$

$$= \frac{a^2}{\pi}\int_0^\pi \frac{\sin^2\theta}{\pi-\theta}\,d\theta = \frac{a^2}{\pi}\int_0^\pi \frac{\sin^2\theta}{\theta}\,d\theta,$$

the base being drawn through a random point on the circumference.

This is also clearly the average area of a triangle formed by taking three random points on the circumference of the circle, since we get the same expression by drawing the base through a fixed point and a random point on the circle.

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13405. (Professor RAMASWAMI AIYAR.) — If a system of similar ellipses have a common centre and all of them touch one of them, the locus of their foci consists of two circles.

Solution by R. F. DAVIS, M.A.; GEORGE HEPPEL, M.A.; *and others.*

Let σ be the focus of an ellipse similar to and concentric with the given ellipse $x^2/a^2 + y^2/b^2 = 1$. Then, if $C\sigma = \rho$ and $\angle \sigma CA = \phi$, the equation to the curve is

$$(x - \rho\cos\phi)^2 + (y - \rho\sin\phi)^2 = e^2\left\{x\cos\phi + y\sin\phi - \rho/e^2\right\}^2,$$

or $$x^2(1 - e^2\cos^2\phi) - 2xye^2\cos\phi\sin\phi + y^2(1 - e^2\sin^2\phi) = \rho^2(1/e^2 - 1),$$

The lines joining the origin with the points of intersection of the two ellipses are given by the homogeneous equation of the second degree

$$\ldots = \rho^2(1/e^2 - 1)(x^2/a^2 + y^2/b^2).$$

If this equation has two equal roots, the curves touch each other. The necessary condition for this will, on reduction, be found to be

$$1 - \frac{\rho^2}{a^2e^2} = \pm\frac{\rho e\sin\phi}{b},$$

or

$$x^2 + y^2 \pm \frac{a^2e^3}{b}y = a^2e^2;$$

which represents two circles passing through the foci. On examination the locus of σ consists of segments of circles at which SS′ subtends an angle $= 2\tan^{-1}a/b$.

13413. (Professor GENESE, M.A.)—The late Professor CATALAN wrote that he had during thirteen years, *à diverses reprises*, unsuccessfully tried to prove directly the following theorem, viz., $a, b, c, \ldots k$ being any unequal quantities,

$$\left\{\frac{1}{a-b} + \frac{1}{a-c} + \frac{1}{a-d} + \ldots + \frac{1}{a-k}\right\}\frac{1}{(a-b)^2(a-c)^2\ldots(a-k)^2}$$
$$+ \left\{\frac{1}{b-a} + \frac{1}{b-c} + \ldots + \frac{1}{b-k}\right\}\frac{1}{(b-a)^2(b-c)^2\ldots(b-k)^2} + \ldots$$
$$+ \left\{\frac{1}{k-a} + \frac{1}{k-b} + \frac{1}{k-c} + \ldots\frac{1}{(k-h)}\right\}\frac{1}{(k-a)^2(k-b)^2\ldots(k-h)^2} \equiv 0:$$

show that the identity may be established either without, or, better, with, the aid of the differential calculus.

Solution by R. F. DAVIS, M.A.

Assume $$\frac{1}{(x-a)^2(x-b)^2\ldots(x-k)^2} \equiv \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2}$$
$$+ \ldots + \frac{K_1}{x-k} + \frac{K_2}{(x-k)^2}.$$

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Then, multiplying up and equating coefficients of like powers of x, it is obvious that $A_1 + B_1 + \ldots + K_1 = \equiv 0$.

We have also

$1 \equiv \{A_1(x-a) + A_2\}\{(x-b)^2(x-c)^2 \ldots (x-k)^2\}$ + terms containing $(x-a)^2$.

Put $x = a$, and we have $A_2 \equiv 1/(a-b)^2(a-c)^2 \ldots (a-k)^2$.

Differentiating with respect to x,

$$0 \equiv A_1 + \{A_1(x-a) + A_2\}\left\{\frac{2}{x-b} + \frac{2}{x-c} + \ldots + \frac{2}{x-k}\right\} + \text{terms containing } (x-a).$$

Put $x=a$, and $-\frac{1}{2}A_1 \equiv \left\{\frac{1}{a-b} + \frac{1}{a-c} + \ldots + \frac{1}{a-k}\right\} \frac{1}{(a-b)^2(a-c)^2 \ldots (a-k)^2}$.

Similar values can be obtained for B_1, $C_1 \ldots K_1$.

---

769. (J. W. Elliott.)—A quantity of corn is to be divided amongst n persons, and is calculated to last a certain time, if each person receives a peck every week; but, during the distribution, it is found that one person dies every week, and the corn in consequence lasts *twice* as long as was expected: required the quantity of corn, and the time it lasted.

Solutions by (1) Professor Zerr; (2) K. S. Putnam.

(1) Let x = number of weeks the corn lasts. Then we get the equation

$$n + (n-1) + (n-2) + \ldots + (n-x) = 1 + 2 + 3 + \ldots + x;$$

or $$n(1+x) = x(1+x);$$ therefore $x = n$.

Then for number of pecks of corn:

n = number used first week, 1 peck was used last week;

n = number of weeks;

therefore $S = \frac{1}{2}n(n+1)$ pecks altogether.

(2) Since the corn lasted twice as long as was expected, there must have been, on the average, but $n/2$ men to receive it. All the persons died. The corn lasted n weeks and there were $n^2/2$ pecks.

If this is true, we must assume that each person died at the middle of the week, so that $n - \frac{1}{2}$ pecks were consumed the first week and $\frac{1}{2}$ peck the last week. Or, if it is objected that the last half peck lasted but half a week, we must assume that each man dying commenced to fall off at the beginning of the week and consumed but $\frac{1}{2}$ peck; or, at least, that the last man did so, received his last kernel at the end of the week, and then died.

If we are to understand that n pecks were issued and consumed the first week, $n-1$ the second, &c., to complete the progression and comply with the conditions, we must have $n+1$ terms, the last being 0. The number of weeks would then be $n+1$ and the quantity of corn $\frac{1}{2}\{n(n+1)\}$ pecks.

In this case we are counting a week in which there is no corn and no person to receive it.

[For another Solution of this Question, see Vol. lxi., p. 117.]

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2487. (The late Rev. R. H. WRIGHT, M.A.)—If a conic be circumscribed about a triangle ABC, and tangents be drawn at A, B, C, and produced to meet so as to form respectively three triangles having the sides of the triangle ABC for their bases, find forms for the bisectors of the angles of the external triangles, and the equations to their circumscribing circles in trilinear coordinates.

Solution by H. J. WOODALL, A.R.C.S.

Let $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ be the circumscribing conic. Then the tangents at A, B, C are

$$m\gamma + n\beta = 0,\ n\alpha + l\gamma = 0,\ m\alpha + l\beta = 0, \text{ respectively.}$$

These meet at

$A'\ (-\alpha/l = \beta/m = \gamma/n)$, $B'\ (\alpha/l = -\beta/m = \gamma/n)$, $C'\ (\alpha/l = \beta/m = -\gamma/n)$.

WHITWORTH has given (*Trilinear Coordinates*, p. 48), an expression for the perpendicular distance of a point from a line $(\lambda\alpha + \mu\beta + \nu\gamma = 0)$, to be

$$p = (\lambda\alpha' + \mu\beta' + \nu\gamma') / \{\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\lambda\mu \cos C\}^{\frac{1}{2}}.$$

Whence we find the equations to the bisectors of the angle of the triangle A'BC to be

$(n\alpha + l\gamma) / \{n^2 + l^2 - 2nl \cos B\}^{\frac{1}{2}} = \pm\ (m\alpha + l\beta) / \{m^2 + l^2 - 2lm \cos C\}^{\frac{1}{2}}$ from A',

$(n\alpha + l\gamma) / \{n^2 + l^2 - 2nl \cos B\}^{\frac{1}{2}} = \pm\ \alpha$ from B,

$(m\alpha + l\beta) / \{m^2 + l^2 - 2lm \cos C\}^{\frac{1}{2}} = \pm\alpha$ from C. Similarly for the other triangles.

Next $a\beta\gamma + b\gamma\alpha + c\alpha\beta = (\lambda\alpha + \mu\beta + \nu\gamma)(a\alpha + b\beta + c\gamma)$ is a circle. This will pass through B and C if μ and $\nu = 0$, through A' if

$$\lambda = (am^2 - bnl - clm)/(al - bm - cn).$$

These substituted above will give the required equation, viz.,

$$(a\beta\gamma + b\gamma\alpha + c\alpha\beta)(al - bm - cn) = (am^2 - bnl - clm)\,\alpha\,(a\alpha + b\beta + c\gamma),$$

and similar equations for the other triangles.

13205. (Rev. J. CULLEN, B.A.)—If ρ be any vector to a curve in space, show that $\int (T\rho)^n T d\rho$ (the integral being taken between finite limits) is a maximum or minimum when $(T\rho)^2 \rho'' = n\rho' V\rho\rho'$, s being the independent variable. Interpret this equation and its first integral.

Solution by the PROPOSER.

The conditions require that $\delta \int (T\rho)^n T d\rho = 0$. Hence

$$d/ds\,[(T\rho)^n \rho'] - \nabla (T\rho)^n = 0 \text{ (TAIT, } Q., \S\S \text{ 496, 450)};$$

therefore $$(T\rho)^n \rho'' - n\,(T\rho)^{n-1} \rho' SU\rho\rho' - n\,(T\rho)^{n-2} \rho = 0;$$

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or $(\mathrm{T}\rho)^2\rho'' = n(\rho'\mathrm{S}\rho\rho' + \rho) = n(\rho'\mathrm{S}\rho\rho' - \rho\rho'^2)(\mathrm{T}\rho' = 1) = n\rho'\,\mathrm{V}\rho\rho'$.....(1).

Taking tensors, we see that the curvature into (radius vector)^2 plus n into the perpendicular vanishes.

Operating on (1) with V. ρ, we obtain

$$(\mathrm{T}\rho)^2\mathrm{V}\rho\rho'' = n\mathrm{V}.\,\rho\rho'\,\mathrm{V}\rho\rho' = n\mathrm{V}.\,(\mathrm{V}\rho\rho' + \mathrm{S}\rho\rho')\,\mathrm{V}\rho\rho' = n\mathrm{S}\rho\rho'\,\mathrm{V}\rho\rho' \quad .....(2).$$

From (1) and (2) we deduce that the tortuosity is zero (*ib.*, § 283).

Operating on (2) with S.$\mathrm{V}\rho\rho'$ and observing that

$$\mathrm{S}(\mathrm{V}\rho\rho'.\mathrm{V}\rho\rho'') = \tfrac{1}{2}\,d/ds\,(\mathrm{V}\rho\rho')^2 \text{ (Hamilton, } \textit{Elements}\text{, p. 665),}$$

and $\mathrm{S}\rho\rho' = -\frac{1}{2}\,d/ds\,(\mathrm{T}\rho)^2$, we have $d/ds\left[\log\left\{\mathrm{T}\mathrm{V}\rho\rho'\,(\mathrm{T}\rho)^n\right\}\right] = 0$;

therefore $\mathrm{T}\mathrm{V}\rho\rho'\,(\mathrm{T}\rho)^n = c$ (a constant scalar).

Hence we conclude that (radius vector)^n varies as the reciprocal of the perpendicular.

2141. (The late Professor Wolstenholme, D.Sc.)—The arc of the curve $x = a(2 - 3\cos\theta + \cos 3\theta)$, $y = 3a\sqrt{2}\,(2\theta - \sin 2\theta)$, from cusp to cusp, is $14a$.

Solution by Professor Zerr.

From the values of x and y, we easily get

$$\frac{dy}{dx} = \frac{2\sqrt{2}\,(1-\cos 2\theta)}{\sin\theta - \sin 3\theta} = \frac{4\sqrt{2}\sin^2\theta}{\sin\theta - \sin 3\theta} = \frac{2\sqrt{2}\sin\theta}{2\sin^2\theta - 1};$$

therefore $$\frac{d\mathrm{S}}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} = \frac{2\sin^2\theta + 1}{2\sin^2\theta - 1}.$$

But $dx = 3a(\sin\theta - \sin 3\theta)\,d\theta = 6a\sin\theta\,(2\sin^2\theta - 1)\,d\theta$;

therefore $$d\mathrm{S} = 6a\sin\theta\,(2\sin^2\theta + 1)\,d\theta;$$

therefore $$\mathrm{S} = 3a\int_0^{\pi}\sin\theta\,(2\sin^2\theta + 1)\,d\theta = 14a.$$

1913. (The late Rev. R. H. Wright, M.A.)—Find the condition in order that a straight line passing through an angular point of the triangle of reference shall be a normal to the conic whose equation is

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0.$$

Solution by H. J. Woodall, A.R.C.S.

Let P $(\alpha', \beta', \gamma')$ be a point of $(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0$; then AP is $\beta/\beta_1 = \gamma/\gamma_1$, tangent at P$s$ $\alpha\,(l/\alpha_1)^{\frac{1}{2}} + \beta\,(m/\beta_1)^{\frac{1}{2}} + \gamma\,(n/\gamma_1)^{\frac{1}{2}} = 0$.

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If these are perpendicular, the coefficients of α, β, γ must satisfy

$$\lambda(\lambda_1 - \mu_1 \cos C - \nu_1 \cos B) + \mu(\ \ ) + \nu(\ \ ) = 0;$$

therefore
$$\gamma_1\{(l/\alpha_1)^{\frac{1}{2}} \cos C + (n/\gamma_1)^{\frac{1}{2}} \cos A - (m/\beta_1)^{\frac{1}{2}}\}$$
$$= \beta_1\{(m/\beta_1)^{\frac{1}{2}} \cos C + (l/\alpha_1)^{\frac{1}{2}} \cos B - (n/\gamma_1)^{\frac{1}{2}}\};$$

and this is the condition that AP should be perpendicular to tangent to the above conic at P.

13345. (Professor HUDSON.)—Find the equation of an ellipsoid referred to the planes of its central circular sections and a central plane at right angles to them. If these are rectangular axes, prove that the squares of the axes are in harmonical progression, and that the equation takes the form
$$\frac{(x+z)^2+y^2}{c^2} + \frac{(x-z)^2+y^2}{a^2} = 2.$$

Solution by Professor ANDERSON, M.A.; H. W. CURJEL, M.A.; *and others.*

Denoting by θ the angle which one of the planes of circular section makes with the greatest axis of the ellipsoid, the equation referred to the new axes is
$$\frac{(x-z)^2 \cos^2\theta}{a^2} + \frac{y^2}{b^2} + \frac{(x+z)^2 \sin^2\theta}{c^2} = 1;$$

or, since
$$\cos^2\theta = \frac{a^2(b^2-c^2)}{b^2(a^2-c^2)},$$

$$\left[(x-z)^2+y^2\right]\frac{b^2-c^2}{a^2-c^2} + \left[(x+z)^2+y^2\right]\frac{a^2-b^2}{a^2-c^2} = b^2.$$

If the new axes are rectangular, $\theta = 45°$; from which it follows that $1/a^2 + 1/c^2 = 2/b^2$, and the given form follows immediately.

13362. (Rev. T. ROACH, M.A.)—If $\Sigma \cos^4 \frac{1}{7}\pi$ denote
$$\cos^4 \tfrac{1}{7}\pi + \cos^4 \tfrac{2}{7}\pi + \cos^4 \tfrac{3}{7}\pi,$$

find the value of $\Sigma \cos^4 \frac{1}{7}\pi$, $\Sigma \sin^4 \frac{1}{7}\pi$, $\Sigma \sec^4 \frac{1}{7}\pi$;
$\Sigma \cos^6 \frac{1}{7}\pi$, $\Sigma \sin^6 \frac{1}{7}\pi$, $\Sigma \tan^6 \frac{1}{7}\pi$, $\Sigma \sec^6 \frac{1}{7}\pi$;
and $\Sigma \cos^8 \frac{1}{7}\pi$, $\Sigma \sin^8 \frac{1}{7}\pi$, $\Sigma \tan^8 \frac{1}{7}\pi$, $\Sigma \sec^8 \frac{1}{7}\pi$.

Solution by Rev. J. CULLEN, B.A.; *the* PROPOSER; *and others.*

Let $f(\cos\theta) = (x-\cos\theta)(x-\cos 2\theta)(x-\cos 3\theta) = 0.$

It is then easy to show that
$$f(\cos^2 \tfrac{1}{7}\pi) = x^3 - \tfrac{5}{4}x^2 + \tfrac{3}{8}x - \tfrac{1}{64} = 0 \quad \ldots\ldots\ldots\ldots\ldots (1).$$

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The equation whose roots are the reciprocals is

$$f(\sec^2 \tfrac{1}{7}\pi) = x^3 - 24x^2 + 80x - 64 = 0 \quad \ldots\ldots\ldots\ldots \quad (2).$$

Changing the sign, and increasing by unity the roots in (1), we have

$$f(\sin^2 \tfrac{1}{7}\pi) = x^3 - \tfrac{7}{4}x^2 + \tfrac{7}{8}x - \tfrac{7}{64} = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (3).$$

Diminishing the roots in (2) by unity, we get,

$$f(\tan^2 \tfrac{1}{7}\pi) = x^3 - 21x^2 + 35x - 7 = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (4).$$

Hence, forming equations whose roots are the second, third, and fourth powers of those in (1), (2), (3), and (4), and equating the coefficients of x^2 in each case, we find

$\Sigma\cos^4 \frac{1}{7}\pi = \frac{13}{16}$, $\Sigma\sin^4 \frac{1}{7}\pi = \frac{41}{16}$, $\Sigma\sec^4 \frac{1}{7}\pi = 416$, $\Sigma\tan^4 \frac{1}{7}\pi = 371$;
$\Sigma\cos^6 \frac{1}{7}\pi = \frac{149}{32}$, $\Sigma\sin^6 \frac{1}{7}\pi = \frac{335}{32}$, $\Sigma\sec^6 \frac{1}{7}\pi = 8256$, $\Sigma\tan^6 \frac{1}{7}\pi = 7077$;
$\Sigma\cos^8 \frac{1}{7}\pi = \frac{1177}{256}$, $\Sigma\sin^8 \frac{1}{7}\pi = \frac{2445}{256}$, $\Sigma\sec^8 \frac{1}{7}\pi = 166400$, $\Sigma\tan^8 \frac{1}{7}\pi = 135779$.

13408. (Professor Chakrivarti.) — A conic always touches four given straight lines; prove that the chord of intersection of the circle, described about any one of the triangles formed by three of these straight lines, with the circle which is the locus of the intersection of two tangents to the conic at right angles to each other, always passes through a fixed point.

Solution by Professor A. Droz-Farny.

On sait qu'un cercle fixe coupe tous les cercles d'un faisceau suivant des droites qui concourent en un point fixe situé sur l'axe radical des cercles du faisceau.

Or les cercles orthotomiques des coniques d'un faisceau tangentiel appartiennent à un faisceau auquel appartiennent aussi les trois circonférences décrites sur les diagonales du quadrilatère donné, comme diamètres. Le point fixe se trouvera donc sur la directrice de la parabole tangente aux quatre droites données.

[Mr. Heppel solves the Question as follows:—

Take AB, AD as axes. Let AB = a, AD = b, and let c be $[u, v]$. Then, applying the formula given in Salmon, Art. 287,

$$\mu^2 E^2 - 2\mu(S + S') + F^2 = 0,$$

a conic touching the four lines reduces to

$$q^2x^2 - 2[pq + 2\mu ab]xy + p^2y^2 + 2abqx + 2abpy + a^2b^2 = 0,$$

where $p = \mu u - a$, $q = \mu v - b$. The centre of this is $[p/\mu, q/\mu]$. The sum of the squares of the semi-axes is $-s\Delta/t^2$; and the director circle reduces to

$$\mu(x^2 + 2xy\cos\omega + y^2) - px - qy - ab\cos\omega = 0.$$

If AB, CD meet in K, the circle ADK is

$$x^2 + 2xy\cos\omega + y^2 - bu/(b - v)\,x - by = 0.$$

As the common chord involves the arbitrary element μ only in the first degree, all values of μ give concurrent chords.]

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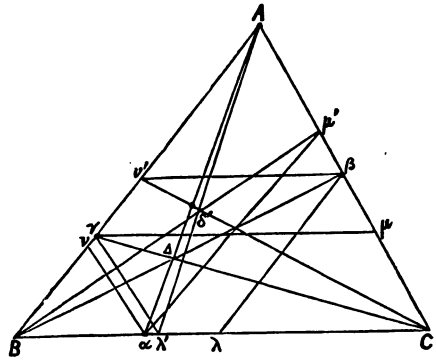
13420. (Capt. H. Brocard. Généralisation de la Quest. 8407.)—Les droites qui joignent les sommets d'un triangle ABC à un point Δ déterminent sur les côtés des points α, β, γ, par chacun desquels on mène des parallèles aux deux autres côtés du triangle donné. L'on obtient ainsi les points (λ', λ), (μ', μ), (ν', ν) sur les côtés a, b, c, en suivant les côtés du triangle respectivement dans l'ordre direct des côtés et dans l'ordre inverse. Cela posé, les triangles $\lambda'\mu'\nu'$, $\lambda\mu\nu$, ainsi déterminés, sont homologiques avec le triangle ABC, et admettent pour centres d'homologie deux points δ' et δ, appelés aussi *point direct* et *point rétrograde* par rapport au point Δ (E. Lemoine). Si le point initial Δ est le point de Lemoine K, δ' et δ deviennent les points de Brocard, ω' et ω. M. de Longchamps a établi que, dans ce cas, les trois triangles $\lambda'\mu'\nu'$, $\lambda\mu\nu$, $\alpha\beta\gamma$ sont équivalents. Démontrer qu'il en est encore ainsi dans le cas général défini par la construction précédente.

Solution by R. F. Davis, M.A.

In the figure $a\nu$, $\beta\lambda$, $\gamma\mu$ are drawn parallel to CA, AB, BC respectively; and $a\mu'$, $\beta\nu'$, $\gamma\lambda'$ are drawn parallel to BA, CB, AC respectively. For simplicity Δ may be taken within ABC so that all segments of the sides are positive, and direction may be disregarded.



Since $A\alpha$, $B\beta$, $C\gamma$ are concurrent, $B\alpha \cdot C\beta \cdot A\gamma = C\alpha \cdot A\beta \cdot B\gamma$. Hence, by parallels, we have $B\nu \cdot C\lambda \cdot A\mu = A\nu \cdot B\lambda \cdot C\mu$, and therefore $A\lambda$, $B\mu$, $C\nu$ are also concurrent. Denoting the sides of ABC by a, b, c respectively,

$$\Delta\lambda\mu\nu = \Delta ABC\left\{1 - A\mu \cdot A\nu/bc - B\nu\ B\lambda/ca - C\lambda \cdot C\mu/ab\right\}.$$

But $$A\mu = A\gamma\,(b/c), \quad A\nu = C\alpha\,(c/a), \text{ \&c.}$$

Hence $$\Delta\lambda\mu\nu = \Delta ABC\left\{1 - A\gamma \cdot C\alpha/ac - B\alpha \cdot A\beta/ab - C\beta \cdot B\gamma/bc\right\}$$

$$= \ldots\ldots\left\{1 - (c - B\gamma)(a - B\alpha)/ac \ldots\ldots\right\}$$

$$= \ldots\ldots\left\{1 - B\gamma \cdot B\alpha/ac \ldots\ldots\right\} = \Delta\alpha\beta\gamma;$$

since $3 = (B\alpha + C\alpha)/a + \ldots + \ldots$. Similarly $\Delta\lambda'\mu'\nu' = \Delta\alpha\beta\gamma$.

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[Professor A. Droz-Farny solves the Question as follows:—

Posons $B\alpha : \alpha C = m : n$; $C\beta : \beta A = m' : n'$; $A\gamma : \gamma B = m'' : n''$. Comme les droites $A\alpha$, $B\beta$, $C\gamma$ se croisent en Δ, on a, d'après le théorème de Ceva, $mm'm'' = nn'n''$. La surface du triangle ABC étant représentée par S, on a, d'après une formule connue,

$$\text{triangle } \alpha\beta\gamma = \frac{mm'm' + 'nn'n''}{(m+n)(m'+n')(m''+n'')}\,S.$$

Comme $\alpha\mu'$ est parallèle à AB, on a $\frac{C\mu'}{A\mu'} = \frac{C\alpha}{B\alpha} = \frac{n}{m}$.

De même $\frac{A\nu'}{B\nu'} = \frac{A\beta}{C\beta} = \frac{n'}{m'}$; $\frac{B\lambda'}{C\lambda'} = \frac{B\gamma}{A\gamma} = \frac{n''}{m''}$.

Donc $$\frac{C\mu'}{A\mu'}\frac{A\nu'}{B\nu'}\frac{B\lambda'}{C\lambda'} = \frac{nn'n''}{mm'm''} = 1;$$

les droites $A\lambda'$, $B\mu'$, $C\nu'$ concourent en un point S'.

En autres les rapports de partage étant les mêmes pour les points μ', ν', λ' que pour α, β, γ, la surface $\mu'\lambda'\nu'$ est équivalente à la surface $\alpha\beta\gamma$. Les démonstrations pour $\lambda\mu\nu$ sont identiques.

On démontre aisément que les coordonnées homogènes du point Δ étant x, y, z on a pour le point S' les coordonnées czx, axy, byz; pour S les coordonnées byx, czy, axz.

Si on prend pour point Δ le point de Lemoine qui a pour coordonnées a, b, c, S' devient ac^2, a^2b, b^2c coincide avec ω'; S devient ab^2, c^2b, a^2c coincide avec ω.]

13295. (I. Arnold.)—If two weights P and Q sustain each other on two inclined planes AB, AC, having a common altitude A, and if a circle be described through the points A, P, Q, and P and Q be connected by a string passing over a pulley E in any part of this circumference, prove that P and Q will still be in equilibrium.

Solution by Rev. J. Cullen, B.A.; H. W. Curjel, M.A.; *and others.*

Since P and Q sustain each other,

$$P \sin\theta = Q \sin\theta_1,$$

Also $$P \sin\theta = T\cos(\theta-\phi),$$

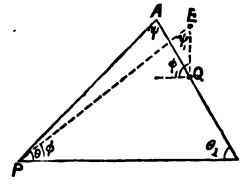
$$Q \sin\theta_1 = T\cos(\phi_1-\theta_1),$$

T being the tension in the string;

$$\therefore\ \cos(\theta-\phi) = \cos(\phi_1-\theta_1),$$

or $$\theta-\phi = \phi_1-\theta_1;$$

$$\therefore\ \psi = \psi_1.$$



Hence the locus of E is a circle passing through P, A, and Q.

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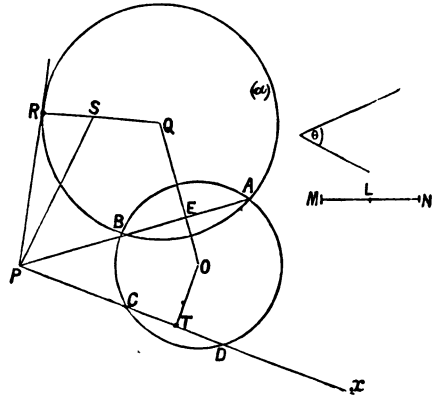
13272. (P. W. Flood.)—Given two chords of a circle, the angle at which they meet, and one of the external segments, describe the circle.

Solution by W. E. Jeffares, B.A.; *the* Proposer; *and others.*

Let AB be one of the chords and PB in its production the given external segment, MN the other segment, and θ the given angle.



From P draw Px making with PA an angle equal to the given angle; through A and B describe any arbitrary circle, and from P draw a tangent PR to it; join Q, its centre, with R; take RS = ML = ½MN; join PS; with P as centre and radius PS, describe a circle cutting Px in T. Now at T and E (the middle point of AB) erect perpendiculars. Where these meet in O is the centre of the required circle; and, with OA or OB as radius, describe a circle cutting Px in C and D. This is the required circle. T is the middle point of CD; therefore

$$PT^2 - CT^2 = PD \cdot PC = PA \cdot PB = PR^2 = PS^2 - RS^2$$
$$= PS^2 - ML^2 = PT^2 - ML^2;$$

$$\therefore \quad CT^2 = ML^2, \text{ and } CT = ML; \quad \therefore \quad CD = MN.$$

In the above solution it is assumed that OA > OT, which is proved as follows:—

$$OB^2 = OP^2 - PA \cdot PB = OP^2 - PR^2 > OP^2 - (PR^2 + RS^2) > OP^2 - PS^2$$
$$> OP^2 - PT^2 > OT^2.$$

13330. (W. C. Stanham.)—Prove, geometrically, that the locus of the foci of parabolas which touch a central conic and its principal axes is the inverse of the evolute of the conic, with respect to the concentric circle through its foci.

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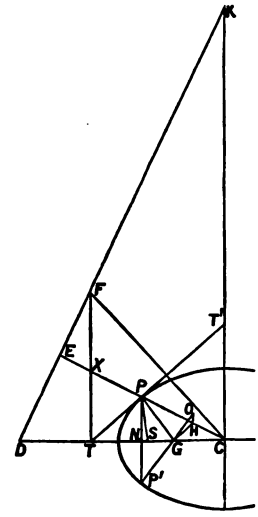
Solution by H. W. CURJEL, M.A.; Prof. MUKHOPADHYAY; *and others.*



Let P be the point of contact of one of the parabolas with the conic. Let TPT′ be the tangent at P, cutting the axes CT, CT′ of the conic in T, T′, the foci S, S′ being on CT. Let normal at P meet CT in G.

Draw PNP′ perpendicular to CT, cutting CT in N and the conic in P′. Draw TF at right angles to CT, cutting CP in X, and CF at right angles to TT′, and KFED at right angles to CP, cutting CT′, CP, CT′ in K, E, D.

Then triangles TPC, CFK are clearly similar.

Hence $DT : TC = DF : FK = XP : PC$
$= TP : PT'$;

therefore D is the point of contact of CT with the parabola; therefore the focus Y of the parabola is the intersection of circle DEC and circle TCT′; and E is on the circle on CF as diameter.

With respect to the circle on SS′ as diameter, G is the inverse of T, for (SS′, GT) is harmonic; therefore the inverse of circle EFCT is GH, which is drawn perpendicular to CF, *i.e.*, parallel to TPT′; therefore inverse of E is H where GH meets CP. Hence the inverse of circle DEC is OH, which is drawn perpendicular to CT.

Again, the inverse of circle CTT′ is a straight line through G perpendicular to the line joining C to the middle point of TT′, and is therefore the normal P′G at P′; therefore inverse of Y is the intersection O of OH and P′G. But it is clear from Quest. 13300, Vol. LXVI., p. 110 (for in the figure of the solution to that question GH is parallel to PM, where H is the point λ in which CM cuts the perpendicular from O to CG), that O is the centre of curvature at P′. Hence the theorem.

13427. (W. C. STANHAM.)—If the pair of equations

$$lx + my + n \pm i\,(l'x + m'y + n') = 0 \quad [\text{where } i = (-1)^{\frac{1}{2}}]$$

are interpreted as representing the locus of the extremity of a line of length $\pm \{(l'x_1 + m'y_1 + n')/n\}$ units of length, drawn from a variable point (x_1, y_1) in the line $lx + my + n = 0$, parallel to a fixed straight line L which neither lies in the plane of (x, y) nor is parallel to it, show that the equations of a pair of imaginary tangents to the circle $x^2 + y^2 = a^2$ from a point (x', y') within it represent two straight lines, and find their angle of inclination to the plane of (x, y) when L is perpendicular to this plane. Mention the limiting values of this angle for limiting values of a and positions of (x', y').

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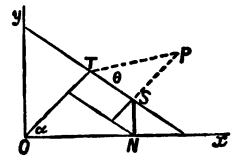
Solution by the PROPOSER.

If the axis of z is taken parallel to L, the locus of the extremity is clearly the intersection of the plane $lx+my+n=0$ with the planes $z=\pm\{(l'x+m'y+n')/n\}$ units of length, and is therefore two straight lines.



Let $x\cos(\alpha\pm i\beta)+y\sin(\alpha\pm i\beta)=a$
be the equations of the tangents. Then, if

$$e^{\beta}+e^{-\beta}=2\sec\phi,$$

this may be written in the form

$$(x\cos\alpha+y\sin\alpha)\sec\phi-a\pm i\,(x\sin\alpha-y\cos\alpha)\tan\phi=0.$$

In the figure, TS is the line $(x\cos\alpha+y\sin\alpha)\sec\phi=a$, S any point in it, and SP drawn according to rule in question; T is the point (x', y'). Let p and q be the numbers of units of length in OT and a. Then

$$\tan\theta=\mathrm{SP/TS}=\mathrm{TS}\tan\phi/q\mathrm{TS},$$

since, if S be (x_1, y_1), $\mathrm{TS}=(x_1\sin\alpha-y_1\cos\alpha)$.

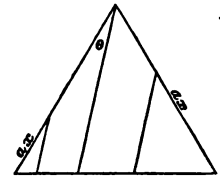
Also $\cos\phi=\mathrm{OT}/a=p/q$; $\quad\therefore\quad\tan\theta=\pm(p^2-q^2)^{\frac{1}{2}}/pq$.

If p is fixed, $\tan\theta$ approaches $1/p$ as q increases. θ approaches 0 when p approaches q. If p approaches 0, and q does not, θ approaches $\pi/2$.

4800. (Professor CAVALLIN, M.A.)—Find the average area of the triangle cut off by a random line from an equilateral triangle.

Solution by H. W. CURJEL, M.A.

The straight line must make an angle less than $\frac{1}{3}\pi$ with some side of the triangle. Hence, if Δ is the area of the given triangle, mean value



$$=\frac{6\Delta}{\pi}\int_0^{\frac{1}{3}\pi}\left\{\frac{\sin\theta}{\sin(\frac{1}{3}\pi+\theta)}\int_0^1\frac{\sin\theta}{\sin(\frac{1}{3}\pi+\theta)}x^2dx\right.$$
$$\left.+\frac{\sin(\frac{1}{3}\pi-\theta)}{\sin(\frac{1}{3}\pi+\theta)}\int_0^1\frac{\sin(\frac{1}{3}\pi-\theta)}{\sin(\frac{1}{3}\pi+\theta)}x^2dx\right\}d\theta$$

$$[\text{for}\quad\sin\theta+\sin(\tfrac{1}{3}\pi-\theta)=\sin(\tfrac{1}{3}\pi+\theta)]$$

$$=\frac{2\Delta}{\pi}\int_0^{\frac{1}{3}\pi}\frac{\sin^2\theta+\sin^2(\frac{1}{3}\pi-\theta)}{\sin^2(\frac{1}{3}\pi+\theta)}d\theta=\frac{2\Delta}{\pi}\int_0^{1/\sqrt{3}}\frac{5y^2-2\sqrt{3}y+3}{(\sqrt{3}+y)^2(1+y^2)}dy$$

(where $y=\tan\theta$)

$$=\frac{2\Delta}{\pi}\int_0^{1/\sqrt{3}}\left\{\frac{6}{(\sqrt{3}+y)^2}-\frac{1}{1+y^2}\right\}dy=\frac{2\Delta}{\pi}\left\{6\left(\frac{1}{\sqrt{3}}-\frac{\sqrt{3}}{4}\right)-\tfrac{1}{6}\pi\right\}$$

$$=\frac{\Delta}{\pi}(\sqrt{3}-\tfrac{1}{3}\pi).$$

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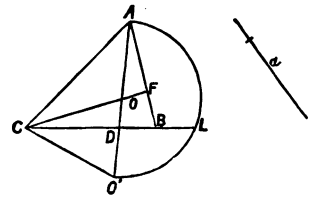
Note on Question 13279.



The Proposer considers Mr. Biddle's solution of this question (Vol. lxvi., p. 105) too intricate. The construction depends on the property BD.DC = AD.DO, and the proof may be given thus. Place the two segments AO, OD of the altitude in a straight line; in AD produced make DO′ = DO; on AO′ describe the semicircle ALO′, and draw DL perpendicular to AO′ to meet the arc. Divide the given side (a) so that the rectangle of the segments = DL² (Euc. II. 14, &c.). On DL mark off DB, BC respectively equal to these segments. then ABC is the required triangle.

The triangles CDO, CDO′ are equal in all respects. Also, as

$$AD.DO' = DL^2 = BD.DC,$$

a circle can go round ACO′B. Hence

$$\angle DCO = O'CB = O'AB = 90° - B;$$

therefore CFA is a right angle, and so is CDA. Hence O is the orthocentre.

13425. (J. J. Walker.)—Show that the vector equation to the surface generated by a transversal meeting the three lines through the terms of the vectors λ, μ, ν and parallel to the vectors α, β, γ respectively is

$$\xi S\alpha\beta\gamma = \Sigma\alpha\left\{(1+h_1)S\mu - h_1 S\nu\right\}V\beta\gamma, \quad \text{with} \quad 1+\Sigma h_1+\Sigma h_2h_3 = 0.$$

Solution by Rev. J. Cullen, B.A.

The equations of the directing lines are $\rho_1 = \lambda + x\alpha$, $\rho_2 = \mu + y\beta$, and $\rho_3 = \nu + z\gamma$. Let ξ be the vector to an arbitrary point on the transversal; then we have

$$\xi = h_1(\rho_2 - \rho_3) + \rho_2 = (1+h_1)\rho_2 - h_1\rho_3 \quad \ldots\ldots(1),$$

also

$$\xi = (1+h_2)\rho_3 - h_2\rho_1 \quad \ldots\ldots(2),$$

$$\xi = (1+h_3)\rho_1 - h_3\rho_2 \quad \ldots\ldots(3).$$

Operating on (1), (2), and (3) with $S\beta\gamma$, $S\gamma\alpha$, and $S\alpha\beta$ respectively, we get

$$S\beta\gamma\xi = \left\{(1+h_1)S\mu - h_1 S\nu\right\}V\beta\gamma, \text{ &c.}$$

Now, applying the known formula $\xi S\alpha\beta\gamma = \Sigma\alpha S\beta\gamma\xi$, we have the stated equation to the surface

$$S(1)(2)(3) = (1+h_1)(1+h_2)(1+h_3)S\rho_1\rho_2\rho_3 - h_1h_2h_3 S\rho_1\rho_2\rho_3 = 0,$$

or

$$(1+\Sigma h_1+\Sigma h_2h_3)S\rho_1\rho_2\rho_3 = 0.$$

Hence the condition, since $\rho_1\rho_2$ and ρ_3 are not in the same plane.

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13399. (Rev. J. CULLEN, B.A.)—Find the general term and the sum to n terms of the series

$$1+2+16+77+391+1952+\ldots.$$

Solution by W. E. JEFFARES, B.A.

Let a convergent series be assumed with the given numbers as coefficients, thus $S = 1+2x+16x^2+77x^3+391x^4+1952x^5+\ldots.$

From the following equations the coefficients of the scale of relation can be found:—

$$f+2g+16h=77, \quad 2f+16g+77h=391, \quad 16f+77g+391h=1952.$$

From these equations we get

$$f=-5, \quad g=1, \quad \text{and} \quad h=5;$$

hence the scale of relation is $1-5x-x^2+5x^3$,

$$S(1-5x-x^2+5x^3) = (1+2x+16x^2+77x^3+391x^4+1952x^5+\ldots)(1-5x-x^2+5x^3).$$

$$= 1-3x+5x^2,$$

$$S = \frac{1-3x+5x^2}{1-5x-x^2+5x^3} = \frac{1-3x+5x^2}{(1-x^2)(1-5x)}.$$

Now resolve $\frac{1-3x+5x^2}{(1-x^2)(1-5x)}$ into its partial fractions; then

$$\frac{1-3x+5x^2}{(1-x^2)(1-5x)} = \frac{1}{8}\left\{\frac{6}{1+x}-\frac{3}{1-x}+\frac{5}{1-5x}\right\}.$$

Hence, by expanding each fraction, and collecting the coefficients of x^{n-1}, we shall get the n^{th} term of the series

$$= \tfrac{1}{8}\left\{5^n-3+(-1)^{n-1}6\right\}x^{n-1};$$

therefore the n^{th} term of the given series is

$$\tfrac{1}{8}\left\{5^n-3+(-1)^{n-1}6\right\}.$$

Now, putting each term of the series in the above form, we have the series in a shape easily summed; for it will consist of three series—

$$\tfrac{1}{8}\left\{(5+5^2+5^3+\ldots+5^n)-3\,(1+1+1+\ldots \text{ to } n \text{ terms}) + 6\,(1-1+1-1-\ldots \text{ to } n \text{ terms})\right\},$$

$$\Sigma = \tfrac{1}{8}\left\{\frac{5^{n+1}-5}{4}-3n+6\right\}, \text{ if } n \text{ is odd},$$

$$\Sigma = \tfrac{1}{8}\left\{\frac{5^{n+1}-5}{4}-3n\right\}, \text{ if } n \text{ is even}.$$

[The PROPOSER solves the question as follows:—We have

$$\left.\begin{matrix} u_{n+2}-5u_{n+1} = -3 \\ u_{n+1}-5u_n = 6 \end{matrix}\right\} n \text{ even} \quad \left.\begin{matrix} = 6 \\ = -3 \end{matrix}\right\} n \text{ odd};$$

$$\therefore \quad u_{n+2}-4u_{n+1}-5u_n = 3,$$

or

$$(E-5)(E+1)u_n = 3.$$

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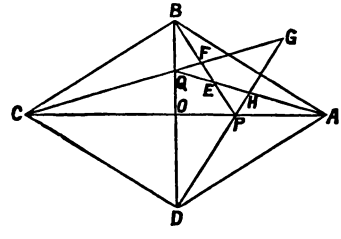
Hence $$u_n = \tfrac{1}{8}\left[1/(E-5) - 1/(E+1)\right](1+0).$$
$$= C5^n + C'(-1)^n - 3/8,$$
$$C = \tfrac{1}{8} \quad \text{and} \quad C' = \tfrac{3}{4}.$$
$$\therefore \quad u^n = \tfrac{1}{8}\left\{5^n + 6(-1)^{n-1} - 3\right\}.$$
$$\text{Sum} = \Sigma u_{n+1} = \tfrac{1}{8}\left\{\tfrac{1}{4}(5^{n+1}) + 3(-1)^{n+1} - 3(n+1)\right\} + C.$$
$$C = \tfrac{19}{32}; \quad \therefore \quad \text{sum} = \tfrac{1}{8}\left\{\tfrac{1}{4}(5^{n+1} - 1) + 3(-1)^{n+1} - 3n + 2\right\}.]$$

13342. (Professor NEUBERG.)—On donne un losange ABCD, et l'on joint un point quelconque P de la diagonale AC aux sommets B et D, puis un point quelconque Q de la diagonale BD aux sommets A et C. Démontrer que les quatre droites ainsi menées forment un quadrilatère inscriptible, et trouver le lieu du centre de la circonférence circonscrite à ce quadrilatère lorsque P est fixe et que Q se meut sur BD.

Solution by H. W. CURJEL, M.A.; *and* W. E. JEFFARES, B.A.

Let the diagonals AC, BD of the rhombus meet in O, and AQ, CQ meet BP in E, F, and PD in H, G.



Then
$$\angle BEQ = \angle EBA + \angle EAB$$
$$= \angle ADP + \angle BCQ$$
$$= \angle FGH;$$
therefore E, F, G, H are concyclic.

Take OA, OB as axes of x and y; let OA, OB, OP, OQ $= a, b, c, m$.

Equations to BP, QA are
$$x/c + y/b = 1, \quad x/a + y/m = 1.$$

Therefore, at E, $x = \dfrac{ca(b-m)}{ab-cm}$; similarly, at F, $x = \dfrac{ca(b-m)}{ab+cm}$;

therefore, at middle point of EF,
$$x = \frac{a^2bc(b-m)}{a^2b^2-c^2m^2}; \quad \therefore \quad y = \frac{(a^2b-c^2m)bm}{a^2b^2-c^2m^2};$$

therefore centre lies in
$$\frac{x}{b} - \frac{y}{c} = \frac{bc^2(a^2+m^2) - ma^2(b^2+c^2)}{c(a^2b^2-c^2m^2)};$$

similarly it lies in the line whose equation is deduced from the last by changing the sign of b.

Therefore, at centre, $x = \dfrac{b^2c(a^2+m^2)}{a^2b^2-c^2m^2}$, $y = \dfrac{a^2m(b^2+c^2)}{a^2b^2-c^2m^2}$;

therefore locus of centre is $a^2cx^2 + xa^2(b^2-c^2) - cb^2y - a^2b^2c = 0$, *i.e.*, an

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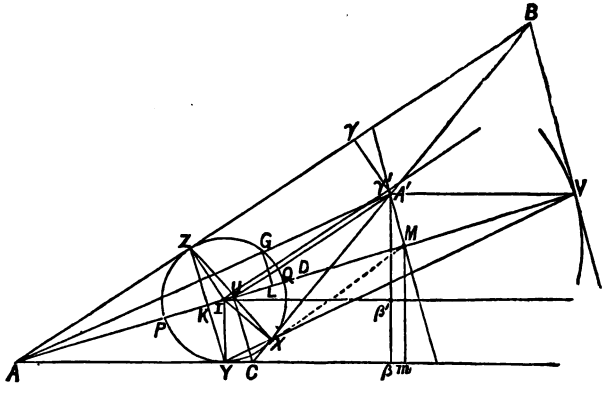
hyperbola with vertices P and the point where the perpendicular to BP through B cuts COA, and asymptotes perpendicular to the sides of the rhombus.

2916 & 4813. (I. H. Turrell.)—To construct a triangle geometrically, having given the vertical angle, radius of inscribed circle, and the centre of gravity of the triangle on the circumference of the inscribed circle.

Solution by C. E. Hillyer.

Let AZB, AYC be two tangents to the given circle XYZ containing ths given angle A. I is the centre of the circle, and AI meets the circumference in P, Q, and YZ in K.

In AQ produced take D so that DQ . DP = DY². Produce AD to M so that DM = ½AK. Describe a circle with centre M cutting XYZ orthogonally and meeting AM in U and V.



Let ZU, YV meet in X; then X shall be on the circumference of given circle, and, if BXC be the tangent at X, ABC shall be the triangle required.

1. X is on the circumference; for U, V are harmonic conjugates with respect to P, Q;

therefore IU . IV = IQ² = IY . IZ; therefore IZ : IU = IV : IY;

therefore triangles IZU, IVY are similar. Therefore ∠IUZ = IYV; therefore I, U, X, Y, are concyclic; ∴ VX . VY = VU . VI = VQ . VP; therefore X is on the circumference. Also CU, BV, and therefore A'M, are perpendicular to AM (see Question 12352) and A'U, A'V are parallel to AB, AC respectively.

2. C is on the circumference; let A'β, A'γ be perpendiculars from A' on AC, AB meeting parallels to AC, AB through I in β', γ'; then

A'β' : IV = A'γ' : IU = KI : IY; ∴ A'β' . A'γ' : IV . IU = KI² : IY²,

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but IV. IU = IY²; ∴ A′β′. A′γ′ = KI² or (A′β−IY)(A′γ−IY) = KI²
also A′β′ + A′γ = 2M*m*.

$$\therefore \quad A'\beta . A'\gamma = 2Mm . IY + KI^2 - IY^2 = 2KI . AM + KI^2 - IY^2$$
$$= 2KI . AD + AK . KI + KI^2 - IY^2 \quad (\because DM = \tfrac{1}{2}AK)$$
$$= 2KI . AD + KY^2 + KI^2 - IY^2 = 2KI . AD.$$

And $$ID^2 = DP . DQ + IY^2 = KY^2 + IY^2,$$

$$\therefore \quad KD^2 = KI^2 + ID^2 + 2KI . ID = 2IY^2 + 2KI . ID$$
$$= 2KI . IA + 2KI . ID = 2KI . AD;$$

$$\therefore \quad A'\beta' . A'\gamma = KD^2.$$

Now, if GL be drawn perpendicular to AI, AL = ⅔AM, and

$$KL = AL - AK = \tfrac{2}{3}AM - \tfrac{2}{3}(AK + DM) = \tfrac{2}{3}KD;$$

therefore KL is a mean proportional between the perpendiculars from G on AB, AC; therefore G is on the circumference.

[Mr. Putnam solves the question as follows:—

Let BAC be the given angle, O_1N_1 the given radius.

Bisect the angle BAC by the indefinite line AO. From any point in that line, as D, let fall upon AB the perpendicular DE. With DE, O_1N_1, and AD find a fourth proportional AO which will establish the centre of the inscribed circle.

To find P, the centre of gravity on the circumference—

Let fall the perpendiculars ON and OF, and draw NF. From G lay off GH = ON; from O lay off OI = NH; take IJ = ⅓IG and erect the perpendicular JP.

(In Vol. LXV. *Reprint*, pp. 44, 45, it is demonstrated that P lies in the perpendicular PJ, when IJ = ⅓IG, if IK.IL = NG². Now IL = IK + 2R,

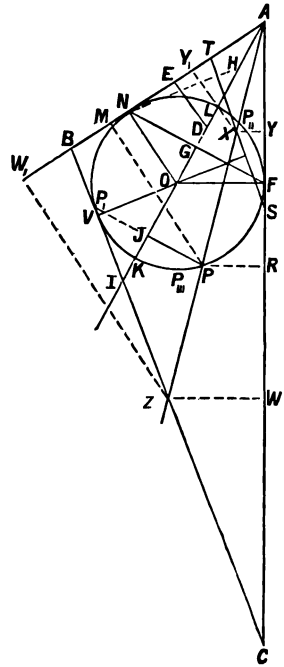
$$\therefore \quad IK(IK + 2R) = NG^2,$$
$$IK^2 + 2R.IK + R^2 = NG^2 + R^2$$
$$= (IK + R)^2.$$

But $$IK + R = OI,$$

and OI = NH, and $NH^2 = NG^2 + GH^2$, and GH = R; ∴ IK.IL = NG².)

To construct the triangle—

From P let fall the perpendiculars PR and PM. Lay off AT = PR and AS = PM. On TS (produced if necessary) let fall perpendicular from O and produce it to V, in the circle. Through V draw the tangent BC perpendicular to radius OV.



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For triangle ABZ = AZC, $ZW.AC = ZW_1.AB$. But $ZW : ZW_1$ :: PR : PM; $\therefore AC.PR = AB.PM$.

Area AXS = AS.XY and area AXT = $AT.XY_1$. But $XY : XY_1$ :: PR : PM. The areas are therefore as AS.PR : AT.PM. But AS = PM and AT = PR. The areas are therefore as PM.PR : PR.PM, or equal.

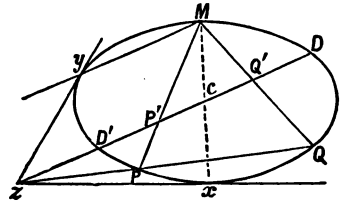
As the bases are in the same line and the vertices at the same point, TX = XS, and the mid-point of any line parallel to TS will be on the line AXPZ; therefore BZ = ZC, as it should, and the construction is correct.

There is a secondary point P_1, showing the location of the centre of gravity when the triangle is constructed on the other side or turned over by revolving about the medial. When B is the given angle, there is a secondary point P_{11}, and with C given, P_{111}. They are only locations of P when the triangle is in different positions.]

[The solution in Vol. LXII., p. 113, assumes that the incentre is in the supposed case necessarily on the median line.]

13422. (A. GORDON.)—Show geometrically how to find a point M on an ellipse such that, P, Q being two points on the same side of any diameter DCD′, MP, MQ shall cut off equal segments on this diameter measured from *c*, the centre. [Extension of one of Professor CATALAN'S.]

Solution by Professor A. DROZ-FARNY.



I Cas : Les points P *et* Q *sont sur l'ellipse.* — Soit M le point cherché; tirons MP et MQ qui coupent DD′ en des points P′ et Q′ tels que CP′ = CQ′ = λ. Lorsque λ varie les paires de droites telles que MP′ et MQ′ engendrent une involution à laquelle appartiennent les droites MD et MD′ et les rayons doubles M*cx* (λ = 0) et M*y* parallèle à DD′ pour λ = ∞. D'après un théorème connu, les droites PQ, DD′, et les tangentes à l'ellipse en *x* et *y*, concourent en un même point *z*. On a de là la solution suivante : PQ et DD′ se croisent en *z*; de *z* on mène à l'ellipse les tangentes *zx* et *zy*; les points M et M′, diamétralement opposés de *x* et de *y*, sont deux solutions du problème.

II Cas : Les points P *et* Q *sont quelconques.*—Soient de nouveau sur le diamètre DD′ deux points P′ et Q′ tels que CP′ = CQ′ = λ; PP′ et QQ′ se coupent en M, et PQ′ et QP′ se coupent en M′. Lorsque λ varie les faisceaux PP′ et QQ′ sont homographiques; il en résulte aisément que M et M′ appartiennent à un même hyperbole qui passe par C pour λ = ∞. Les points d'intersection de l'hyperbole et de l'ellipse, dont deux au moins sont réels, sont les points cherchés.

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13372. (R. KNOWLES, B.A.)—If PP′ be a diameter of a central conic, and a tangent from P meets a chord P′R in Q, prove that the tangent from R bisects PQ.

Solution by Rev. J. CULLEN, B.A.; W. J. GREENSTREET, M.A.; *and others.*

Take the tangent and normal for axes. Then the conic is

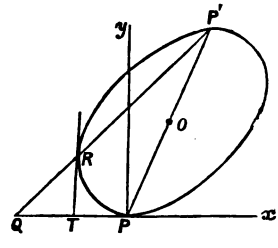
$$ax^2+2hxy+by^2+2fy=0.$$

The chord P′R is $y=mx-(m\alpha+\gamma)$,

where $\alpha=2hf/(ab-h^2)$,

$\gamma=2af/(ab-h^2)$;

$\therefore$ PQ $=(m\alpha+\gamma)/m$.



Tangent at R (x_1, y_1) cuts axis of x in T.

$$\text{PT}=-fy_1/(ax_1+hy_1),$$

where

$$x_1=(m\alpha+\gamma)(h+bm)/(a+2hm+bm^2),$$

and $y_1=-(m\alpha+\gamma)(a+hm)/(a+2hm+bm^2)$; $\therefore$ $2\text{PT}=(m\alpha+\gamma)/m=\text{PQ}$.

[Professor A. DROZ-FARNY solves the question as follows:—Soient O le centre de la conique et T le point d'intersection de la tangente en R avec PQ. T étant le pôle de PR, cette droite sera divisée en A en parties égales par le diamètre TO. Comme PA = AR et PO = OP′, TO est parallèle à OP′ et par conséquent PT = TQ.]

13348. (Professor H. L. ORCHARD, M.A., B.Sc.)—Sum to n terms the series $s=3+5+14+36+88+208+\&c.$

Solution by Rev. D. THOMAS, M.A.; W. J. GREENSTREET, M.A.; *and others.*

The sum of n terms of s is the same as 3 plus the sum of $n-1$ terms of

$$1.5+2.7+4.9+8.11+16.13+\&c.,$$

which sum is $-1+2^{n-1}(2n-1)$ because the nth term is $2^{n-1}(2n+3)$.

13241. (J. J. BARNIVILLE, B.A.)—Prove that

$$\frac{1}{1+1}+\frac{1}{2+1}+\frac{1}{5+1}+\frac{1}{13+1}+\ldots=\frac{\sqrt{5}}{2},$$

$$\frac{1}{1+i}+\frac{1}{3+i}+\frac{1}{8+i}+\ldots=1-\frac{i}{2}(\sqrt{5}-1).$$

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Solution by Professors SANJÁNA, KRISHMACHANDRA DE, *and others.*

By the usual transformation into a continued fraction,

$$\text{the first series} = \frac{1}{2-}\,\frac{4}{5-}\,\frac{9}{9-}\,\frac{36}{20-}\,\frac{196}{49-}\cdots = \frac{1}{2-}\,\frac{4}{5-x},$$

where $$x = \frac{1}{1-}\,\frac{1}{5-}\,\frac{1}{1-}\,\frac{1}{5-}\cdots = \frac{5-\sqrt{5}}{2}, \text{ evidently.}$$

Thus the first series $= 1\Big/\left(2-\frac{8}{5+\sqrt{5}}\right) = \frac{5+\sqrt{5}}{2+2\sqrt{5}} = \frac{\sqrt{5}}{2}$.

By rationalization, the second series

$$= \frac{1-i}{2} + \frac{3-i}{10} + \frac{8-i}{65} + \frac{21-i}{442} + \cdots$$

$$= \tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{8}+\tfrac{1}{8}-\tfrac{1}{18}+\tfrac{1}{18}\cdots - i\left(\frac{1}{1.2}+\frac{1}{2.5}+\frac{1}{5.13}\cdots\right)$$

$$= 1 - i\,\frac{\sqrt{5}-1}{2}.$$ (See solution of Quest. 13118, Vol. LXV., p. 117.)

13341. (R. CHARTRES.)—If three points be taken at random in the sides of a triangle, one in each side, prove that the mean value of each $= \frac{1}{4}\Delta$; also, if two points be taken at random in the sides, the mean value of the triangle AFE cut off $= \frac{1}{4}\Delta$.

Solution by H. W. CURJEL, M.A.; Professor MUKHOPADHYAY; *and others.*

If two points are fixed, the mean value is clearly obtained by placing the remaining point at the middle point of its range, *i.e.*, of the side in which it is. If this be done with all three points in succession, the triangle will be divided into four equal triangles. Therefore mean value of each of the triangles (including AFE) $= \frac{1}{4}\Delta$.

4909. (Professor GENESE, M.A.)—Find functions f and ϕ of two variables, so that $\alpha = f(xy)$, $\beta = \phi(xy)$, $x = f(\alpha\beta)$, $y = \phi(\alpha\beta)$.

Solution by H. W. CURJEL, M.A.

An obvious form of the functions f and ϕ is

$$f(x, y) = \psi^{-1}\{a\psi(x) + b(1+a)\chi(y)\},$$

$$\phi(x, y) = \chi^{-1}\left\{\frac{(1-a)}{b}\psi(x) - a\chi(y)\right\},$$

where ψ and χ are arbitrary functions. This appears to be the most general form.

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3761. (W. Siverly.)—One end of a rod, whose length is equal to the diameter of a spherical shell, is passed through a hole in the shell and made to touch every point of the interior surface: find the surface described by the other end of the rod.

Solution by H. W. Curjel, M.A.; Professor G. B. M. Zerr, M.A.; *and others.*

Let $(x+a)^2+y^2+z^2=a^2$ be the equation to the sphere. The generating curve is evidently the cardioid $r=2a(1-\cos\theta)$; therefore the equation to the surface described is $(x^2+y^2+z^2)^2+4ax(x^2+y^2+z^2)=4a^2(y^2+z^2)$, the part described by the end of the rod corresponding to positive values of x.

13466. (Professor M. J. M. Hill, M.A., D.Sc., F.R.S.)—Give proofs of (1) the 9th; (2) the 16th, with the 14th as corollary; (3) the 22nd; (4) the 23rd propositions of the Fifth Book of Euclid, based on Euclid's test for equal ratios, to replace his demonstrations, which are based on his test for unequal ratios; and hence show that all the propositions in the Sixth Book of Euclid can be proved without any reference direct or indirect to his test for unequal ratios.

Solution by the Proposer.

(*Euc.* v. 9) Part 1.—If A : C = B : C, then A = B.

Suppose, if possible, that A and B are unequal; then one of them is greater than the other. Let A be greater than B; then it is possible to find an integer n so that a multiple of C lies between nA and nB.

Suppose nA > tC > nB. Then nA > tC, nB < tC. But, if A : C = B : C, and nA > tC, then must nB > tC. But nB < tC. Hence A and B are not unequal. Therefore A = B.

Part 2.—If C : A = C : B, then A = B.

As in Part 1, it may be proved that, if A be greater than B, integers n and t can be found such that tC < nA, whilst tC > nB.

Now, if C : A = C : B, and tC < nA, then tC < nB. But tC > nB. Hence A is not greater than B. In like manner, A is not less than B. Therefore A = B.

(*Euc.* v. 16.)—If A : B = C : D, and A, B, C, D are all magnitudes of the same kind, then A : C = B : D.

Suppose r, s are integers, such that rA > sC; then either there is a multiple of B between rA and sC or not. If not, it is always possible to find an integer n such that there is a multiple of B between nrA and nsC.

Let tB lie between these; therefore nrA > tB > nsC; because nrA > tB, and A : B = C : D, therefore nrC > tD; therefore snrC > stD; but, because tB > nsC, therefore rtB > rnsC. Now snrC = rnsC; therefore rtB > stD; therefore rB > sD.

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Hence, if $rA > sC$, then $rB > sD$ ..... (I.).

In like manner, if $rA < sC$, $rB < sD$..... (II.),

the proof being the same as the above, with all the signs of inequality reversed.

Again, since $A : B = C : D$, $\therefore$ $B : A = D : C$.

Hence the conclusions which can be deduced from (I.) and (II.) by interchanging A and B, C and D, hold.

Then (I.) becomes, if $rB > sD$, then $rA > sC$ ..... (III.).

and (II.) becomes, if $rB < sD$, then $rA < sC$ ..... (IV.),

From these it follows that, if $rA = sC$, then $rB = sD$ ..... (V.),

and, if $rB = sD$, then $rA = sC$..... (VI.).

For, suppose that $rA = sC$, but $rB \neq sD$; let $rB > sD$.

Now, by (III.), since $rB > sD$, it follows that $rA > sC$, which is contrary to hypothesis.

In like manner, by (IV.), rB cannot be less than sD. Hence $rB = sD$.

Hence (V.) holds. In like manner (VI.) holds.

$$\therefore \quad A : C = B : D.$$

Corollary (*Euc.* v. 14). — If $A : B = C : D$, then $A \gtreqless C$ according as $B \gtreqless D$.

This is already proved in the work of the proposition, being the particular case in which $r = s = 1$.

(*Euc.* v. 22.)—If $A : B = T : U$, and $B : C = U : V$, prove that $A : C = T : V$.

If $rA > sC$, and no multiple of B lie between rA and sC, then it is always possible to find an integer n so that there will be a multiple of B between nrA and nsC.

Let $nrA > tB > nsC$. Because $nrA > tB$, and $A : B = T : U$, therefore $nrT > tU$. Because $tB > nsC$, and $B : C = U : V$, therefore $tU > nsV$. Therefore $nrT > tU > nsV$; therefore $nrT > nsV$; therefore $rT > sV$.

Hence, if $rA > sC$, $rT > sV$..... (I.).

In like manner, if $rA < sC$, $rT < sV$ ..... (II.).

This may be shown by rewriting the above work, and reversing all the signs of inequality.

In like manner, if $rT > sV$, it will follow that $rA > sC$ ..... (III.),

and, if $rT < sV$, then $rA < sC$ ..... (IV.).

This may be shown by writing the given proportions

$$T : U = A : B, \quad \text{and} \quad U : V = B : C,$$

and making the corresponding changes.

It remains only to consider the case where $rA = sC$. In this case rT must $= sV$, for, if not, it follows, by (III.) and (IV.), that rA is not equal to sC, which is contrary to the hypothesis.

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Hence, if $rA = sC$, then $rT = sV$ ..... (V.),

and conversely, if $rT = sV$, then $rA = sC$..... (VI.).

$\therefore$ A : C = T : V.

(*Euc.* v. 23.)—If A : B = U : T, and B : C = V : U, to prove that

A : C = V : T.

If $rA > sC$, take n some integer such that there lies some multiple of B between nrA and nsC.

Let $nrA > tB > nsC$. Because $nrA > tB$, and A : B = U : T, therefore $nrU > tT$. Since $tB > nsC$, and B : C = V : U ; therefore $tV > nsU$.

From the two inequalities $nrU > tT$, $tV > nsU$ it follows that $nrsU > stT$, and $rtV > rnsU$.

Now $nrsU = rnsU$; therefore $rtV > stT$; therefore $t(rV) > t(sT)$; therefore $rV > sT$.

Hence, if $rA > sC$, then $rV > sT$ ..... (I.).

In like manner, by reversing the signs of inequality, if

$rA < sC$, then $rV < sT$..... (II.).

Also, reversing the proportions, T : U = B : A, U : V = C : B. These might also be obtained from the original proportions by interchanging A with T, B with U, C with V.

Making these changes in (I.) and (II.), and also interchanging r and s, which can be done, since r, s are any positive integers, it follows from (I.) that, if

$sT > rV$, then $sC > rA$ ..... (III.),

which is the converse of (II.) ; and from (II.), if

$sT < rV$, then $sC < rA$ ..... (IV.),

which is the converse of (I.).

It remains only to consider the case if $rA = sC$. In this case rV must be equal to sT, for, if not, it follows from (III.) and (IV.) that rA is not equal to sC.

Hence, if $rA = sC$, then $rV = sT$ ..... (V.),

and, if $rV = sT$, then $rA = sC$ ..... (VI.).

$\therefore$ A : C = V : T.

It follows from the above that the Test for Unequal Ratios can be dispensed with, so far as the propositions in the Sixth Book of Euclid are concerned. PEYRARD's Text (1814) of Euclid is employed. The propositions of the Fifth Book which are quoted in the Sixth Book are 7, 9, 11, 12, 15, 16, 18, 22, 24. To these 23 may be added if we desire to give a complete account of the compounding of ratios. Of the propositions above mentioned, 7, 11, 12, 15 do not depend on the Test for Unequal Ratios; whilst 16 and 18 depend on 14, and 24 depends on 18 and 22. But 14 is really a particular case of 16. Consequently all that is necessary is to give proofs of 9, 16, 22, 23, which are independent of the Test for Unequal Ratios.

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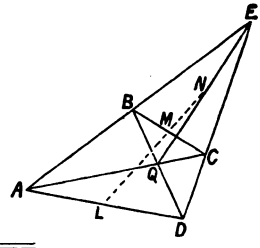
13478. (Morgan Brierley.)—ABCD is a given quadrilateral with the opposite sides AB, DC produced to meet in E; draw EQ to the intersection of the diagonals AC, BD. If a line LM be drawn through the middle of the sides AD, BC, it will pass through the middle N of EQ.

Solution by E. P. Barrett, B.A.; H. W. Curjel, M.A.; *and others.*



Consider the quadrilateral EAQD (re-entrant angle at Q).

The three diagonals are AD, BC, and EQ.

Therefore (Lachlan, *Modern Geometry*, Ex. 1, p. 23) L, M, and N, the middle points of these, are collinear.

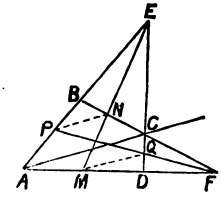
13423. (H. W. Curjel, M.A.)—If the opposite sides AB, DC of a quadrilateral meet in E, and AD, BC in F, and any straight lines EMN, PFQ are drawn through E and F, cutting AD, BC and AB, DC in M, N and P, Q, show that PN, MQ cut on AC, and PM, NQ on BD.

Solution (1) *by* W. C. Stanham, *and others*; (2) *by* Professor A. Droz-Farny.

(1) Let N_1, N_2 denote two other positions of N, and M_1, M_2 corresponding positions of M, PQ remaining unaltered in position. Then

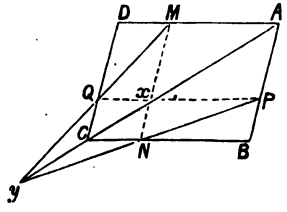
$$E(FNN_1N_2) = E(FMM_1M_2);$$

$$\therefore\ P(FNN_1N_2) = Q(FMM_1M_2).$$



Since these equal pencils have a common ray (PQ), the intersections of corresponding rays are collinear. And, when ENM coincides with EBA and ECD, it is clear that A and C are points on the line of collinearity, which must therefore be AC. Hence PN and QM cut on AC, and similarly PM and NQ on BD.

(2) Effectuons une projection centrale de la figure de manierè à ce que EF tombe à l'infini; la question devient la suivante:—Soit x un point pris dans l'intérieur d'un parallélogramme ABCD. La parallèle à AB menée par x coupe les côtés AD et BC en M et N et la parallèle par le même point à AD coupe les côtés AB et CD en P et Q. Les diagonales qui ne passent pas par x des quatre petits parallélogrammes ainsi obtenus se croisent par paires sur les diagonales AC et BD du parallélogramme donné. En effet supposons



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que MQ et PN rencontrent AC en deux points distincts y et y'. Dans les triangles ADC et ABC on aura en vertu du théorème de MÉNÉLAUS :

$$AM \cdot DQ \cdot yC = DM \cdot CQ \cdot yA, \quad BN \cdot PA \cdot y'C = CN \cdot BP \cdot y'A ;$$

d'où, par division, $yC : y'C = yA : y'A$;

donc les points y et y' coincident.

13402. (Professor C. COCHEZ.)—On donne deux axes rectangulaires Ox, Oy. Une parabole reste tangente à Oy en O et passe par un point A donné sur Ox, ayant pour abscisse $2a$. Trouver et construire (1) le lieu du point de contact d'une tangente à la parabole, parallèle à Ox ; (2) l'enveloppe de la directrice de la parabole ; (3) le lieu du foyer de la parabole ; (4) le lieu du sommet de la même parabole.

Solution by Rev. J. CULLEN, B.A.

Since tangent and normal are axes, the equation is of the form

$$(x + \lambda y)^2 + 2gx = 0,$$

under the stated condition

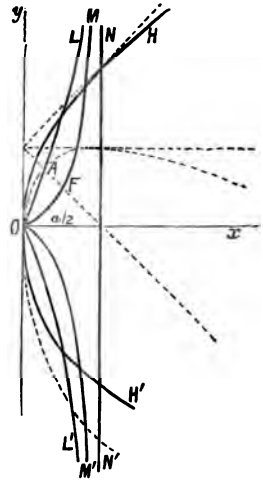
$$4a^2 + 4ga = 0 ;$$

$$\therefore \quad (x + \lambda y)^2 - 2ax = 0 \quad \ldots\ldots\ldots(\alpha).$$

With ordinary notation,

$$A = 0, \quad B = -a^2, \quad C = 0,$$

$$F = -a\lambda, \quad G = a\lambda^2, \quad H = 0.$$



(1) The condition that $y - \beta = 0$ should touch (α) is $2\lambda\beta = a$; therefore $x = a/2$. Hence the required locus is the line NN′.

(2) The directrix is

$$Gx + Fy = (A + B)/2,$$

or $$2\lambda^2 x - 2\lambda y + a = 0 ;$$

therefore envelope is

$$y^2 - 2ax = 0,$$

the parabola HOH′.

(3) The focus is determined by

$$Fx + Gy = H,$$

and $$Gx - Fy = (A - B)/2,$$

or $$x = a/2\,(1 + \lambda^2), \quad y = a/2\lambda\,(1 + \lambda^2), \quad i.e., \quad 2\,(x^2 + y^2)\,x = ay^2,$$

the cissoid MOM′.

(4) The axis is $x + \lambda y = a/(1 + \lambda^2)$. Hence the coordinates of the vertex are $x = a/2\,(1 + \lambda^2)^2, \quad y = a\,(1 + 2\lambda^2)/2\lambda\,(1 + \lambda^2)^2,$

or $$2x\,(x^2 + y^2)^2 - a\,(8x^4 + y^4 - 8ax^3) = 0.$$

The curve LOL′.—NN′ is an asymptote to this curve and the cissoid ; it

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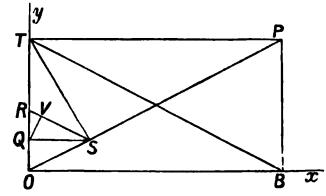
also determines the latus rectum of HOH′. The broken-lined portion of the figure represents (a) its axis, directrix, and tangent, when $\lambda = 1$.

[Mr. CURJEL solves the question as follows:—Let P be the point of contact of the tangent PT parallel to Ox; then the parallel to the axis through P bisects OA. Let the parallel to the axis through T, where PT, Oy meet, cut Ox in B. Then $OB = \frac{1}{2}OA = \frac{1}{2}a$; and $\angle$ PBO is a right angle, since $\angle$ OTP is a right angle; therefore locus of P is $2x = a$. Also the directrix is clearly at right angles to BT, and passes through T; therefore its envelope is a parabola, focus B and vertex O, *i.e.*, $y^2 = 2ax$. Draw TS perpendicular to OP; then S is the focus; therefore its locus is $r = \frac{1}{2}a \tan\theta \sin\theta$, where $\angle POB = \theta$, *i.e.*, $2x(x^2+y^2) = ay^2$. Draw SQ perpendicular to OT, and SR parallel to TB, and QV perpendicular to SR. Then V is the vertex, and its coordinates are $x = \frac{1}{2}a \sin^4\theta$, $y = \frac{1}{2}a \sin^3\theta\,(1/\cos\theta + \cos\theta)$;



$$\therefore \quad 4(x^2+y^2)\cos^2\theta/a^2 = 4\sin^6\theta - 3\sin^8\theta;$$

therefore locus is $a(y^2+4x^2)^2 = 2x(x^2+y^2+2ax)^2$,

i.e., $a(y^4+8x^4) = 2x(x^4+y^4+2x^2y^2+4a^2x^2)$.]

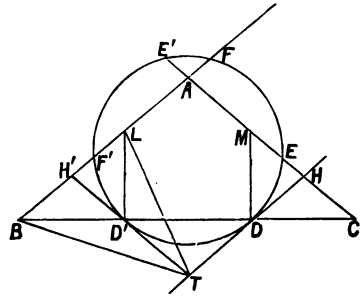
13388. (W. J. GREENSTREET, M.A.) — A conic cuts the sides of a triangle ABC in D, D′; E, E′; F, F′, respectively. The tangents at D, D′ meet AB, AC in K, K′; L is the fourth harmonic of B with respect to F, F′; M that of C with respect to E, E′. Show D′L, DM, KK′ are concurrent.

Solution by H. W. CURJEL, M.A.

Project the points in which KK′ meets the conic into the circles.

Then DEE′FF′D′ becomes a circle such that the tangents at D and D′ are parallel to AB and AC.

Let these tangents cut AB, AC in H′, H and each other in T. Then TL is the polar of B. Therefore T(DD′, BL) is harmonic; therefore H′ is the middle point of BL.



Therefore

$H'L^2 = H'F \cdot H'F' = H'D'^2$;

therefore $H'D' = H'L = H'B$; therefore $\angle BD'L$ is a right angle; similarly $\angle MDC$ is a right angle; therefore D′L, DM are parallel, *i.e.*, cut on KK′.

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13284. (Professor DICKSON.)—Find the general term in the series 1, 3, 10, 35, 126, 462, 1716, 6435, 24310, ... which plays a remarkable part in some recent theorems in the theory of regular polygons.

Solution by R. W. D. CHRISTIE.

We see at once that any term $a_n \times 3\,[(n-2)/n] = a_{n+1}$; therefore the nth term of the series is the product of $n-1$ terms of the series

$$\frac{4n-2}{2}\ \frac{4n-6}{3}\ \frac{4n-10}{4}\ \frac{4n-14}{5},\ \&c.$$

Thus the nth term is $\dfrac{2^{n-1}\,.\,2n-1\,.\,2n-3\,.\,2n-5\,\ldots\,3}{n\,!}$

Ex. gr., the 7th term is $\dfrac{2^6.13.11.9.7.5.3}{5040} = 1716.$

13430. (Professor SYLVESTER.)—If the digits r in number of any integer N read from left to right be multiplied repeatedly by the first r terms of the recurring series 1, 4, 3, −1, −4, −3, 1, 4, 3, −1, −4, −3, show that, if the sum of these products be divisible by 13, so N will be, and not otherwise.

Solution by G. N. BATES.

By reducing $\frac{1}{13}$ to a decimal, the following congruences to mod 13 are seen to be true:—

$$10^{6n} \equiv 1;\quad 10^{6n+1} \equiv -3;\quad 10^{6n+2} \equiv -4;\quad 10^{6n+3} \equiv -1;$$
$$10^{6n+4} \equiv 3;\quad 10^{6n+5} \equiv 4.$$

Let N be a number having $(6n+1)$ digits. Then

$$a_r\left\{10^{r-1}-1\right\} + a_{r-1}\left\{10^{r-2}-4\right\} + \ldots \equiv 0,\quad \text{mod}\,(13),$$

i.e., $a_r a_{r-1} \ldots a_1$ is divisible by 13 if

$$a_r + 4a_{r-1} + 3a_{r-2} - a_{r-3} - \ldots$$

is, and not otherwise.

If N has not $(6n+1)$ digits, sufficient 0's can be added at the right-hand end to make it so, and these will not affect its divisibility by 13.

[Mr. CURJEL solves the Question as follows:—

Since $10^3 \equiv -1$ and $10^6 \equiv 1$, mod 13, it will be sufficient to show that $a + 4b + 3c\,(= m)$ and $100a + 10b + c\,(= n)$ are neither or both divisible by 13, which is clear, since $(4m+n) = 13\,(8a+2b+c).$]

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13446. (Editor.)—Prove that
$4\{(a^2-b^2)cd+(c^2-d^2)ab\}^2+\{(a^2-b^2)(c^2-d^2)-4abcd\}^2=(a^2+b^2)^2(c^2+d^2)^2$.

Solution by Rev. T. Wiggins, B.A.; L. E. Reay, B.A.; *and many others.*

Exp. $= 4\{a^2cd-b^2cd+c^2ab-d^2ab\}^2+(a^2c^2-b^2c^2-a^2d^2+b^2d^2-4abcd)^2$
$= 4\{(ac-bd)(ad+bc)\}^2+\{(ac-bd)^2-(ad+bc)^2\}^2$
$= \{(ac-bd)^2+(ad+bc)^2\}^2 = (a^2c^2+b^2d^2+b^2c^2+a^2d^2)^2$
$= (a^2+b^2)^2(c^2+d^2)^2$.

[Mr. Davis and Mr. Cullen solve the Question as follows:—
Put $a^2=(a^2+b^2)\cos^2\theta$ and $c^2=(c^2+d^2)\cos^2\phi$;
then transform algebraically the relation

$$\sin^2 2(\theta+\phi)+\cos^2 2(\theta+\phi)=1.]$$

13462. (Professor Crofton, F.R.S.)—Prove that 10^5+10 is divisible by 9^3+1.

Solution by Rev. T. Wiggins, B.A.; *and* Morgan Brierley.

$$\frac{10^5+10}{9^3+1}=\frac{10^4+1}{9^2-9+1}=\frac{10^4+1}{10^2-3\times10+3}$$
$$=10^2+3\times10+6+\frac{9\times10-17}{10^2-3\times10+3}$$
$$=10^2+3\times10+6+\frac{10^2-10-17}{10^2-3\times10+3}$$
$$=10^2+3\times10+6+\frac{10^2-3\times10+3}{10^2-3\times10+3}=10^2+3\times10+7.$$

13468. (Professor Neuberg.) — On considère les quadriques qui passent par une conique donnée Σ et par deux points donnés A, B. Trouver le lieu du centre et le lieu des extrémités du diamètre qui passe par le centre de Σ.

Solution by Rev. J. Cullen, B.A.

Take the centre of Σ for the origin. Then the quadric is of the form

$$\Sigma+z(\lambda P_1+\mu P_2+P_3)=0 \quad \ldots\ldots\ldots\ldots (1),$$

where P_1, P_2, and P_3 are planes depending on the positions of A and B. Also (1) may be written in the form

$$S_1+\lambda S_2+\mu S_3=0 \quad \ldots\ldots\ldots\ldots (2),$$

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and, since the equations for the centre are

$$\frac{dS_1}{dx}+\lambda\frac{dS_2}{dx}+\mu\frac{dS_3}{dx}=0 \quad \text{.....(3),}$$

$$\frac{dS_1}{dy}+\lambda\frac{dS_2}{dy}+\mu\frac{dS_3}{dy}=0 \quad \text{..... (4),}$$

$$\frac{dS_1}{dz}+\lambda\frac{dS_2}{dz}+\mu\frac{dS_3}{dz}=0 \quad \text{..... (5),}$$

we have

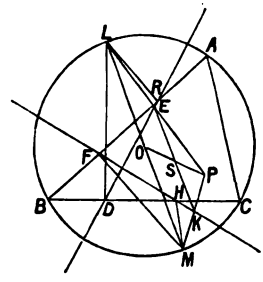
$$\begin{vmatrix} \frac{dS_1}{dx}, & \frac{dS_2}{dx}, & \frac{dS_3}{dx} \\ \frac{dS_1}{dy}, & \frac{dS_2}{dy}, & \frac{dS_3}{dy} \\ \frac{dS_1}{dz}, & \frac{dS_2}{dz}, & \frac{dS_3}{dz} \end{vmatrix} = 0 = \Delta \text{ (suppose),}$$

the locus of the centre.

If α, β, γ are the coordinates of the centre, the diameter through the origin is $x/\alpha = y/\beta = z/\gamma$. They also satisfy (3), (4), and (5), so that $\lambda = \Delta_2/\Delta_1$, $\mu = \Delta_3/\Delta_1$, where Δ_1, Δ_2, and Δ_3 are minors of Δ and quadratic functions of α, β, γ. Hence, by means of the equations of the diameter, $\Delta_1 S_1 + \Delta_2 S_2 + \Delta_3 S_3 = 0$ and $\Delta = 0$ can be expressed in terms of α, and therefore the eliminant is the locus of the extremities of the diameter.

13484. (W. E. Jeffares, B.A.)—If ABC is a triangle, and O its circumcentre, through O draw any diameter LOM intersecting the circumcircle in L and M; find the locus of the intersection of the Simson-lines corresponding to L and M.

Solution by W. S. Cooney, B.A.; *and the* Proposer.



If P be the orthocentre of ABC, then the Simson-lines DR and FH bisect PL and PM respectively at E and K. Therefore

$$EK = \tfrac{1}{2}LM = R,$$

and OP is bisected at S, the centre of nine-point circle. DR and HF intersect at right angles, for

$$\angle KFM = CBM = BLD = BRD.$$

Therefore locus of intersection is nine-point circle (centre S).

13394. (H. Orfeur.)—Besides the circle, there is another curve possessing the property radius of curvature = radius vector. How may it be analytically described? Show that it is a particular involute of the involute of a circle.

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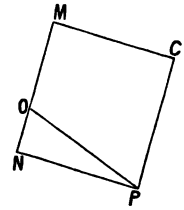
Solution (1) *by* H. W. CURJEL, M.A.; (2) *by* W. C. STANHAM, *and* Rev. J. CULLEN, B.A.

(1) If the radius of curvature = radius vector,

$$r\,(dr/dp) = r; \qquad \therefore \quad r = p + a \quad \ldots\ldots (1)$$

(this is a circle, if $a = 0$).



Let NP be the tangent at point P on the curve, and C the centre of curvature. Through the origin O draw NOM parallel to PC, and through C draw CM parallel to PN.

Then $\quad OM = CP - ON = r - p = a;$

therefore CM is the tangent at M to a circle, centre O. Therefore the curve (P) is an involute of an involute of a circle, such that, when the centre of curvature C of (P) at P lies on the circle, P bisects OC.

The equation (1) may be written

$$d\theta = \frac{dr}{(2ar - a^2)^{\frac{1}{2}}} - \frac{a\,dr}{r\,(2ar - a^2)^{\frac{1}{2}}};$$

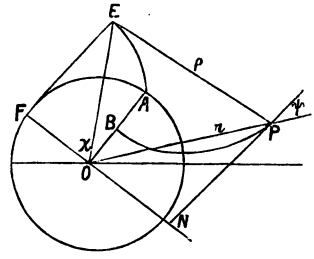
$$\therefore \quad \theta = \frac{(2ar - a^2)^{\frac{1}{2}}}{a} - 2\tan^{-1}\frac{(2ar - a^2)^{\frac{1}{2}}}{a},$$

if $\theta = 0$, when $r = a/2$.

(2) With usual notation,

$$\rho = r\,(dr/dp) = r;$$

$$\therefore \quad \rho = r = p + c.$$



Let PB denote the curve in which c is not zero, as in the case of a circle. Let E be the centre of curvature at P, PN the tangent, O the origin. Draw EF and FON perpendicular to EP and PN. Let $\psi = \angle OPN$, $\chi = \angle FOE = \frac{1}{4}\pi + \frac{1}{2}\psi$. Then OF must be equal to c, and therefore EF envelopes the circle AF, and PE is the tangent at E to the evolute of PB, and EF is the normal at E to this evolute. Therefore PB is a particular involute of the involute of a circle, the particular involute being determined from the fact that, if the curve AE starts from a fixed point A in the circle, we have at this point $\chi = 0$, and therefore $\psi = -\frac{1}{2}\pi$, and

$$r = p + c = r\sin\psi + c; \qquad \therefore \quad r = \tfrac{1}{2}c; \qquad \therefore \quad OB = \tfrac{1}{2}OA.$$

To describe this curve, the usual method of unwinding a string from its evolute (which is described in the same manner from the circle) may be adopted; the free end of the string starting from B. Or we may find any number of points on the curve by proceeding as follows:—Draw a tangent FE to the circle, take E such that FE = arc AF, draw EP perpendicular to EF, and OP so that $\angle EOP = \angle EOF$; then P is a point on the curve PB.

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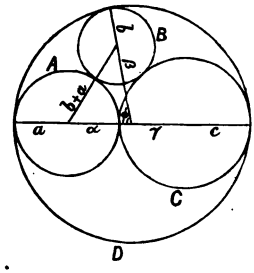
13409. (Professor SARKAR.)—Three circles, touching each other externally, are all touched by a fourth circle including them all. If a, b, c be the radii of the three internal circles, and α, β, γ the distances of their centres from that of the external circle, respectively, prove that

$$2(\beta\gamma/bc + \gamma\alpha/ca + \alpha\beta/ab) = 4 + \alpha^2/a^2 + \beta^2/b^2 + \gamma^2/c^2.$$

Solution by Rev. J. CULLEN, B.A.; *and* W. E. JEFFARES, B.A.



Let R be the radius of the external circle D. Then $a + \alpha = b + \beta = c + \gamma = R$.

The locus of the centre of a circle touching A externally and D internally is

$$(b + a)^2 = r^2 + \alpha^2 - 2r\alpha\cos\phi,$$

or $(2a - r + \alpha)^2 = r^2 + \alpha^2 - 2r\alpha\cos\phi$.

Therefore $a(R - r) = \alpha r \sin^2 \frac{1}{2}\phi$;

putting $r = \beta$, we have

$$\sin^2 \tfrac{1}{2}\phi_1 = ab/\alpha\beta.$$

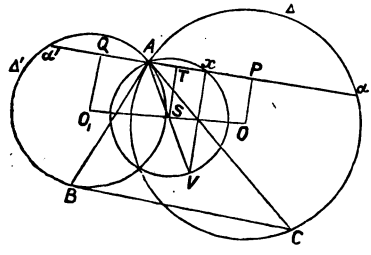
Similarly, $\sin^2 \frac{1}{2}\phi_2 = ac/\alpha\gamma$,

and $\sin^2(\phi_2 - \phi_1)/2 = bc/\beta\gamma$.

The elimination of ϕ_1 and ϕ_2 between these equations gives the above relation, ϕ_1 and ϕ_2 being the angles between α and β, α and γ.

13464. (Professor A. DROZ-FARNY.)—Dans un triangle ABC on considère la circonférence Δ tangente en A à AB et qui passe par C et la circonférence Δ' tangente en A à AC et qui passe par B. Une transversale variable menée par A coupe Δ en α et Δ' en α'. Quel est le lieu du point milieu de $\alpha\alpha'$?

Solution by W. E. JEFFARES, B.A.; H. W. CURJEL, M.A.; *and others.*



Let O, O_1 be the centres of Δ, Δ', and S the middle point of OO_1; O, O_1, S are fixed points. From O, O_1, and S let fall perpendiculars OP, O_1Q, and ST on any transversal $\alpha\alpha'$ through A. Now T is the middle point of PQ; therefore

$$PA - AQ = 2AT,$$

and $\alpha A - \alpha' Q = 4AT$.

Join AS; produce AS to V so that SV = SA; and from V let fall Vx perpendicular to $\alpha\alpha'$. Since

$$AV = 2AS, \quad \therefore \; Ax = 2AT, \quad \text{and} \quad \alpha A - \alpha' Q = 2Ax;$$

therefore x is the middle point of $\alpha\alpha'$. Now, AV is a fixed line and AxV

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a right angle, the locus of x is a circle on AV as diameter. Hence, to find the locus, we have only to bisect OO_1, the line joining the centres in S. Join SA; and, with centre S, radius SA, describe the circle.

13424. (R. F. Davis, M.A.)—(1) If E be the centre of mean position for multiples l, m, n, at A, B, C, respectively, and ET the tangent to the circumcircle of ABC, prove that

$$l\mathrm{PA}^2 + m\mathrm{PB}^2 + n\mathrm{PC}^2 = (l+m+n)(\mathrm{PE}^2 - \mathrm{ET}^2).$$

Deduce that the equation $l\mathrm{PA}^2 + \ldots + \ldots = 0$ represents generally a circle cutting orthogonally the circumcircle, the trilinear coordinates of whose centre are $(l/a,\ m/b,\ n/c)$. When $l+m+n=0$ the above locus degenerates into a diameter of the circumcircle; as, for example,

$$(b^2-c^2)\,\mathrm{PA}^2 + \ldots + \ldots = 0$$

represents the join of the circum- and ortho-centres upon which lie, *inter alia*, the special points $\mathrm{PA}^2 : \mathrm{PB}^2 : \mathrm{PC}^2 = a^2 : b^2 : c^2$, or $\cot \mathrm{A} : \cot \mathrm{B} : \cot \mathrm{C}$.

(2) If ABC be a triangle, prove that the locus represented by

$$a^2(b^2-c^2)\,\mathrm{PA}^2 + b^2(c^2-a^2)\,\mathrm{PB}^2 + c^2(a^2-b^2)\,\mathrm{PC}^2 = 0$$

is the join of the circumcentre and the symmedian point ("diamètre de Brocard"); and upon it lie (*inter alia*) the point-pairs defined by

$\mathrm{PA} : \mathrm{PB} : \mathrm{PC} = \operatorname{cosec} \mathrm{A} : \operatorname{cosec} \mathrm{B} : \operatorname{cosec} \mathrm{C}$ (equilateral pedal triangles),

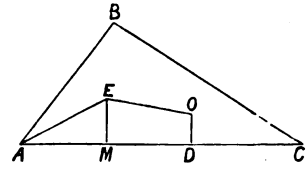
or $\quad = \cot \mathrm{A} : \cot \mathrm{B} : \cot \mathrm{C}$

or $\quad = (a^2 - b^2c^2/a^2) : \ldots : \ldots$ &c., &c.

Solution by G. N. Bates, B.A.; *and* W. E. Jeffares, B.A.

E is the point

$$\alpha = \frac{2\Delta}{l+m+n} \cdot \frac{l}{a}, \text{ &c.}$$



$$\mathrm{AE}^2 = \frac{\beta^2 + \gamma^2 + 2\beta\gamma\cos \mathrm{A}}{\sin^2 \mathrm{A}}$$

$$= \left\{1/(l+m+n)^2\right\} \left\{c^2m^2 + b^2n^2 + 2bcmn\cos \mathrm{A}\right\};$$

$$\therefore \quad \Sigma l\mathrm{AE}^2 = \left\{1/(l+m+n)\right\} \Sigma mna^2,$$

$$\mathrm{OE}^2 = \mathrm{AO}^2 + \mathrm{AE}^2 - b\,(\mathrm{AE}^2 - \mathrm{EM}^2)^{\frac{1}{2}} + 2\mathrm{R}\cos \mathrm{B} \,.\, \mathrm{EM};$$

$$\therefore \quad (l+m+n)^2(\mathrm{R}^2 - \mathrm{OE}^2) = \Sigma mna^2 = (l+m+n)\,\Sigma l\mathrm{AE}^2.$$

Now $\quad \Sigma l\mathrm{PA}^2 = \Sigma l\mathrm{AE}^2 + (l+m+n)\,\mathrm{EP}^2 = (l+m+n)(\mathrm{EP}^2 + \mathrm{R}^2 - \mathrm{OE}^2)$

$$= (l+m+n)(\mathrm{EP}^2 - \mathrm{ET}^2),$$

$l\mathrm{PA}^2 + m\mathrm{PB}^2 + n\mathrm{PC}^2 = 0$ represents the locus $\mathrm{EP}^2 + \mathrm{R}^2 = \mathrm{OE}^2$,

i.e., a circle centre E $(l/a,\ m/b,\ n/c)$ cutting circumcircle orthogonally.

When $l+m+n=0$, E is at infinity, and locus degenerates into diameter of circumcircle.

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Therefore $(b^2-c^2)\,PA^2+(c^2-a^2)\,PB^2+(a^2-b^2)\,PC^2=0$

is a line on which (*inter alia*) the following points lie :—

$PA=PB=PC$ (circumcentre),

$PA:PB:PC::\cos A:\cos B:\cos C$ (orthocentre),

$PA^2:PB^2:PC^2::a^2:b^2:c^2$, $PA^2:PB^2:PC^2::\cot A:\cot B:\cot C$.

Similarly $a^2(b^2-c^2)\,PA^2+\ldots+\quad=0$

represents a line through O on which (*inter alia*) lie the points given by

$PA:PB:PC::\{2(b^2+c^2)-a^2\}/a^2:\qquad:$ (K)

$::$ cosec A : cosec B : cosec C (equilateral pedal Δ's) $::\cot A:\cot B:\cot C$

and $PA^2:PB^2:PC^2::\{a^2-(b^2c^2/a^2)\}:\qquad:\qquad$ &c.

Note on Quest. 13424, *by the* Acting Editor.

(1) Mr. Jeffares writes:—"In these remarks I give you the work whereby I find that Mr. Davis must have made a mistake in the ratios in the last line of this Question (Vol. lxvii., p. 113).

In case of finding the pair-points, we find the middle points of the intervals cutting the sides of a triangle internally and externally in the given ratios: these three points are in a line whose equation is

$$l^2a\alpha+m^2b\beta+n^2c\gamma=0,$$

when $l:m:n$ are the three ratios and the line on which the pair-points lie is a perpendicular from circumcentre on this line. See Question 13363 given by Mr. Davis himself.

According to this, the equation of the line joining the middle points is

$$\frac{(a^4-b^2c^2)^2}{a^3}\alpha+\frac{(b^4-c^2a^2)^2}{b^3}\beta+\frac{(c^4+a^2b^2)^2}{c^3}\gamma=0,$$

for Mr. Davis' ratios $(a^2-b^2c^2/a^2)$: &c.

Now this line should be perpendicular to

$$a\sin(B-C)+\beta\sin(C-A)+\gamma\sin(A-B)=0,$$

or, taking sides instead of angles,

$$\frac{b^2-c^2}{a}\alpha+\frac{c^2-a^2}{b}\beta+\frac{(a^2-b^2)}{c}\gamma=0.$$

Now, if two lines

$$l\alpha+m\beta+n\gamma=0\quad\text{and}\quad l'\alpha+m'\beta+n'\gamma=0$$

be perpendicular to each other, then

$$ll'+mm'+nn'-(mn'+m'n)\cos A-(nl'+n'l)\cos B-(lm'+l'm)\cos C=0;$$

then, by using the coefficients of the equations

$$\frac{(a^4-b^2c^2)^2}{a^3}\alpha+\frac{(b^4-c^2a^2)^2}{b^3}\beta+\frac{(c^4-a^2b^2)^2}{c^3}\gamma=0,$$

$$\frac{(b^2-c^2)}{a}\alpha+\frac{(c^2-a^2)}{b}\beta+\frac{a^2-b^2}{c}\gamma=0,$$

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$$\frac{(a^4-b^2c^2)^2(b^2-c^2)}{a^4}+\frac{(b^4-c^2a^2)^2(c^2-a^2)}{b^4}+\frac{(c^4-a^2b^2)^2(a^2-b^2)}{c^4}$$
$$-\left[\frac{(b^4-c^2a^2)^2(a^2-b^2)}{b^3c}+\frac{(c^4-a^2b^2)^2(c^2-a^2)}{bc^3}\right]\cos A$$
$$-\left[\frac{(c^4-a^2b^2)^2(b^2-c^2)}{c^3a}+\frac{(a^4-b^2c^2)^2(a^2-b^2)}{ca^3}\right]\cos B$$
$$-\left[\frac{(a^4-b^2c^2)^2(c^2-a^2)}{a^3b}+\frac{(b^4-c^2a^2)^2(b^2-c^2)}{ab^3}\right]\cos C;$$

or
$$\frac{(a^4-b^2c^2)^2}{a^4}\left[\frac{b^2-c^2}{a}-\frac{a^2-b^2}{c}\cos B-\frac{c^2-a^2}{b}\cos C\right]$$
$$+\frac{(b^4-c^2a^2)^2}{b^3}\left[\frac{c^2-a^2}{b}-\frac{a^2-b^2}{c}\cos A-\frac{b^2-c^2}{a}\cos C\right]$$
$$+\frac{(c^4-a^2b^2)^2}{c^3}\left[\frac{a^2-b^2}{c}-\frac{c^2-a^2}{b}\cos A-\frac{b^2-c^2}{a}\cos B\right].$$

Now $\frac{b^2-c^2}{a}-\frac{a^2-b^2}{c}\cos B-\frac{c^2-a^2}{b}\cos C$

$$=\frac{b^2-c^2}{a}-\frac{(a^2-b^2)}{c}\frac{(c^2+a^2-b^2)}{2ca}-\frac{(c^2-a^2)}{b}\frac{(a^2+b^2-c^2)}{2ab}$$
$$=\frac{b^2-c^2}{a}-\frac{b^2c^2(a^2-b^2)+(a^2-b^2)^2b^2+b^2c^2(c^2-a^2)-(c^2-a^2)^2c^2}{2ab^2c^2}$$
$$=\frac{2b^2c^2(b^2-c^2)+c^2(c^2-a^2)^2+b^2c^2(b^2-c^2)-b^2(a^2-b^2)^2}{2ab^2c^2}$$
$$=\frac{3b^2c^2(b^2-c^2)+c^2(c^2-a^2)^2-b^2(a^2-b^2)^2}{2ab^2c^2}$$
$$=\frac{(b^2-c^2)\left[2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4\right]}{2ab^2c^2}$$
$$=\frac{(b^2-c^2)\times 16\Delta^2}{2ab^2c^2},$$ where Δ is area of triangle.

Also for $\frac{c^2-a^2}{b}-\frac{a^2-b^2}{c}\cos A-\frac{b^2-c^2}{a}\cos C$ we have $\frac{(c^2-a^2)\times 16\Delta^2}{2bc^2a^2}$, and for $\frac{a^2-b^2}{c}-\frac{c^2-a^2}{b}\cos A-\frac{b^2-c^2}{a}\cos B$ we have $\frac{(a^2-b^2)\times 16\Delta^2}{2ca^2b^2}$. The whole reduces to $\frac{16\Delta^2}{2a^4b^4c^4}\left[(a^4-b^2c^2)^2b^2c^2(b^2-c^2)+(b^4-c^2a^2)^2c^2a^2(c^2-a^2)+(c^4-a^2b^2)^2a^2b^2(a^2-b^2)\right]$, which does not vanish unless the triangle is equilateral.

I now proceed to show that the ratios I gave satisfy the condition. In this case, however, the pair-points are *imaginary* for one of the quantities $\left[(a^4-b^2c^2)^{\frac{1}{2}}/a\right]$, $\left[(b^4-c^2a^2)^{\frac{1}{2}}/b\right]$, $\left[(c^4-a^2b^2)^{\frac{1}{2}}\right]/c$ is imaginary, and therefore two of the ratios imaginary and the pair-points are imaginary. We can see from this that the equation of the line of centres is that of the line joining the Brocard points and must only cut one side of the triangle externally.

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The equation of the centres being $l^2a\alpha + m^2b\beta + n^2c\gamma = 0$, therefore for the ratios $\left[(a^4-b^2c^2)^{\frac{1}{2}}/a\right] : \left[(b^4-c^2a^2)^{\frac{1}{2}}/b\right] : \left[(c^4-a^2b^2)^{\frac{1}{2}}/c\right]$ it is

$$\left[(a^4-b^2c^2)/a\right]\alpha + \left[(b^4-c^2a^2)/b\right]\beta + \left[(c^4-a^2b^2)/c\right]\gamma = 0,$$

and that of the line joining circumcentre with symmedian point is

$$(b^2-c^2)\,a/a + (c^2-a^2)\,\beta/b + (a^2-b^2)\,\gamma/c = 0;$$

hence the condition that they should be perpendicular is

$$\begin{aligned}
&\frac{(a^4-b^2c^2)(b^2-c^2)}{a^2} + \frac{(b^4-c^2a^2)(c^2-a^2)}{b^2} + \frac{(c^4-a^2b^2)(a^2-b^2)}{c^2} \\
&\quad - \left[\frac{(b^4-c^2a^2)(a^2-b^2)}{bc} + \frac{(c^4-a^2b^2)(c^2-a^2)}{bc}\right]\cos A \\
&\quad - \left[\frac{(c^4-a^2b^2)(b^2-c^2)}{ca} + \frac{(a^4-b^2c^2)(a^2-b^2)}{ca}\right]\cos B \\
&\quad - \left[\frac{(a^4-b^2c^2)(c^2-a^2)}{ab} + \frac{(b^4-c^2a^2)(b^2-c^2)}{ab}\right]\cos C \\
&= (a^4-b^2c^2)\left[\frac{b^2-c^2}{a^2} - \frac{a^2-b^2}{ca}\cos B - \frac{c^2-a^2}{ab}\cos C\right] \\
&\quad + (b^4-c^2a^2)\left[\frac{c^2-a^2}{b^2} - \frac{a^2-b^2}{bc}\cos A - \frac{b^2-c^2}{ab}\cos C\right] \\
&\quad + (c^4-a^2b^2)\left[\frac{a^2-b^2}{c^2} - \frac{b^2-c^2}{ca}\cos B - \frac{(c^2-a^2)}{bc}\cos A\right].
\end{aligned}$$

Now $\dfrac{(b^2-c^2)}{a^2} - \dfrac{a^2-b^2}{ca}\cos B - \dfrac{c^2-a^2}{ab}\cos C = \dfrac{1}{a}\cdot\dfrac{(b^2-c^2)\times 16\Delta^2}{2ab^2c^2}$ as above.

Similarly, $\dfrac{c^2-a^2}{b^2} - \dfrac{a^2-b^2}{bc}\cos A - \dfrac{b^2-c^2}{ab}\cos C = \dfrac{c^2-a^2}{2a^2b^2c^2}\times 16\Delta^2$,

and $\dfrac{a^2-b^2}{c^2} - \dfrac{b^2-c^2}{ca}\cos B - \dfrac{c^2-a^2}{bc}\cos A = \dfrac{a^2-b^2}{2a^2b^2c^2}\times 16\Delta^2$;

therefore

$$\begin{aligned}
&\frac{(a^4-b^2c^2)(b^2-c^2)}{a^2} + \frac{(b^4-c^2a^2)(c^2-a^2)}{b^2} + \frac{(c^4-a^2b^2)(a^2-b^2)}{c^2} \\
&\quad - \left[\frac{(b^4-c^2a^2)(a^2-b^2)}{bc} + \frac{(c^4-a^2b^2)(c^2-a^2)}{bc}\right]\cos A \\
&\quad - \left[\frac{(c^4-a^2b^2)(b^2-c^2)}{ca} + \frac{(a^4-b^2c^2)(a^2-b^2)}{ca}\right]\cos B \\
&\quad - \left[\frac{(a^4-b^2c^2)(c^2-a^2)}{ab} + \frac{(b^4-c^2a^2)(b^2-c^2)}{ab}\right]\cos A \\
&= \frac{16\Delta^2}{2a^2b^2c^2}\left[(a^4-b^2c^2)(b^2-c^2) + (b^4-c^2a^2)(c^2-a^2) + (c^4-a^2b^2)(a^2-b^2)\right] = 0,
\end{aligned}$$

since the portion within the brackets is cipher.

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(2) Mr. BATES writes on seeing the foregoing:—I agree with Mr. JEFFARES that there is a misprint in the last line of the question. This consists in omission of $PA^2 : PB^2 : PC^2$ before the ratios given. The connexion between the point $PA^2 : PB^2 : PC^2 :: a^2 - b^2c^2/a^2$ and the BROCARD line joining $\Omega\Omega'$ given in trilinears $(a^4 - b^2c^2)\,a/a + \ldots + = 0$, I cannot see, and it seems difficult to express $\Omega\Omega'$ as an equation in x, y, z, where $x = PA$.

The equation $\Sigma a^2 (b^2 - c^2)\, PA^2 = 0$ can be shown to represent the line OK independently of earlier part of question, by means of the equations of transformation $x^2 \sin^2 A = \beta^2 + \gamma^2 + 2\beta\gamma \cos A$, &c.,

which reduce it to

$$(a\alpha + b\beta + c\gamma)\left[(b^2 - c^2)\,a/a + (c^2 - a^2)\,\beta/b + (a^2 - b^2)\,\gamma/c\right] = 0,$$

i.e., a constant and the ordinary trilinear equation for OK. Similarly $\Sigma (b^2 - c^2)\, x^2 = 0$ breaks up into constant and ordinary equation for OH.

(3) Mr. JEFFARES further says:—I am glad to see that Mr. BATES, to whom you sent my remarks about the ratios in the last part of Mr. DAVIS's question, agrees that it must be either a misprint or an oversight when writing out the question.

I think your contributor has mistaken my proof, or he would have seen that I did not mean that the BROCARD points were pair-points, but that the line joining Ω, Ω' is perpendicular to the line on which the pair-points lie, that is, the line of centres of the coaxal circles whose intersections are the pair-points. (In the case of the line $\Omega\Omega'$ being the line of centres, the pair-points are imaginary.)

The equation of the line of centres of the three coaxal circles which determine the pair-points is given by Mr. DAVIS in Quest. 13363,

$$l^2 a\alpha + m^2 b\beta + n^2 c\gamma = 0,$$

where $pA : pB : pC :: l : m : n$. If in this equation you put for l^2, m^2, n^2 the ratios $a^2 - (b^2c^2/a^2) : b^2 - (c^2a^2/b^2) : c^2 - (a^2b^2/c^2)$ it will be the equation of the line $\Omega\Omega'$, and therefore $\Omega\Omega'$ is perpendicular to the radical axis of the three circles described on the segments of the sides of the triangle contained between the points in which BC, CA, and AB are cut internally and externally in the ratio of $m : n$, $n : l$, and $l : m$, and from the similarity of these ratios to the coefficients of α, β, γ in the equations to well-known lines that he gave the ratios in Quest. 13424.

13347. (Professor STEGGALL.)—A number n of given forces act at a point O: the resultant, by the polygon construction, is OP. Show that in the construction of *all* the $n!$ polygons by which this resultant can be obtained there are $n \cdot 2^{n-1}$ straight lines.

Solution by H. W. CURJEL, M.A.

Let the number of straight lines $= F_n$. If we have drawn the F_{n-1} lines, where there are $(n-1)$ forces, and add another force, each of the F_{n-1} lines can have a new position, obtained by moving it a distance representing the new force in magnitude and direction. Also the number

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of angular points of the polygons of the $(n-1)$ forces is 2^{n-1}, for these are clearly ${}^{n-1}C_r$ positions of the $(r+1)^{th}$ angular point, counting from the origin (which is the first). Hence there are 2^{n-1} positions of the line representing the new force. Therefore

$$F_n = 2F_{n-1} + 2^{n-1}; \quad \text{therefore } F_n = n.2^{n-1} + a.2^n;$$

but $F_1 = 1$ and $F_2 = 4$; therefore $a = 0$; therefore $F_n = n.2^{n-1}$.

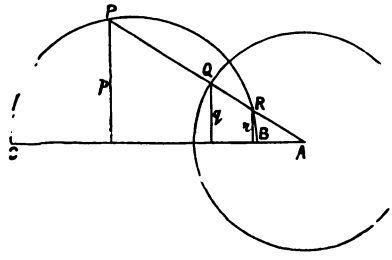
13373. (P. W. Flood.)—From a given point in the diameter produced of a given semicircle, draw a line cutting the circumference in two points such that, if perpendiculars be drawn from them on the diameter, their rectangle will be given.

Solution by H. W. Curjel, M.A.; Professor A. Droz-Farny; *and others.*

Let A be the fixed point on the diameter BC of the ⊙PRBC. Draw a circle with centre A cutting ⊙PRBC orthogonally, and on that circle find Q such that the square of its distance q from BC = the given rectangle. Let AQ cut ⊙PRBC in P and R. Then

$$AR.AP = AQ^2;$$
$$\therefore \quad pr = q^2,$$

where p and r are the distances of P and R from BC.



2501. (N'Importe.)—Find the equation whose roots are the differences of the roots of the equation $(a, b, c, d, e)(x, 1)^4 = 0$.

Solution by H. J. Woodall, A.R.C.S.

Write the given equation thus,

$$a(x^4 - p_1x^3 + p_2x^2 - p_3x + p_4) \equiv (a, b, c, d, e \barwedge x, 1)^4 = 0.$$

The new equation in the differences of these roots will be

$$y^{12} - P_1y^{10} + P_2y^8 - P_3y^6 + P_4y^4 - P_5y^2 + P_6 = 0,$$

where we have to determine P_1, P_2, &c.

Let $s_r = \alpha^r + \beta^r + \gamma^r + \delta^r$; $S_r = (\alpha-\beta)^{2r} + (\alpha-\gamma)^{2r} + \dots$.

Then we have the following equations

$$S_1 - P_1 = 0, \quad S_2 - S_1P_1 + 2P_2 = 0, \text{ &c.},$$

whence we have $P_1 = S$, $P_2 = \frac{1}{2}(S_1P_1 - S_2)$, &c.

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The S's may be obtained by TODHUNTER's method (*Equations*, p. 175), viz.,

$$S_m = ns_{2m} - 2ms_1 s_{2m-1} + \frac{2m(2m-1)}{1.2} s_2 s_{2m-1} - \ldots + (-1)^m \tfrac{1}{2} \frac{(2m)!}{(m!)^2} (s_m)^2.$$

The s's may be taken out of a table (*e.g.*, MEYER HIRSCH), in terms of $p-s$. Making the calculations, we get

$$S_1 = 4s_2 - (s_1)^2 = 3p_1^2 - 8p_2,$$

$$S_2 = 4s_4 - 4s_1 s_3 + 3(s_2)^2 = 3p_1^4 - 16p_1^2 p_2 + 20p_2^2 + 4p_1 p_3 - 16p_4,$$

$$S_3 = 4s_6 - 6s_1 s_5 + 15 s_2 s_4 - 10 (s_3)^2 = 3p_1^6 - 24p_1^4 p_2 + 66p_1^2 p_2^2 - 68p_2^3 - 6p_1^3 p_3 + 42p_1 p_2 p_3 - 78p_3^2 - 54p_1^2 p_4 + 144p_2 p_4,$$

$$\begin{aligned} S_4 = {} & 4s_8 - 8s_1 s_7 + 28 s_2 s_6 - 56 s_3 s_5 + 35 (s_4)^2 = 3p_1^8 - 32p_1^6 p_2 + 136 p_1^4 p_2^2 \\ & - 288 p_1^2 p_2^3 + 260 p_2^4 - 24 p_1^5 p_3 + 208 p_1^3 p_2 p_3 - 408 p_1 p_2^2 p_3 - 204 p_1^2 p_3^2 \\ & + 640 p_2 p_3^2 - 144 p_1^4 p_4 + 768 p_1^2 p_2 p_4 - 928 p_2^2 p_4 - 288 p_1 p_3 p_4 + 576 p_4^2, \end{aligned}$$

$$\begin{aligned} S_5 = {} & 4s_{10} - 10 s_1 s_9 + 45 s_2 s_8 - 120 s_3 s_7 + 210 s_4 s_6 - 126 (s_5)^2 \\ = {} & 3p_1^{10} - 40 p_1^8 p_2 + 230 p_1^6 p_2^2 - 740 p_1^4 p_2^4 + 1330 p_1^2 p_2^4 - 1028 p_2^5 - 50 p_1^7 p_3 \\ & + 570 p_1^5 p_2 p_3 - 2060 p_1^3 p_2^2 p_3 + 2390 p_1 p_2^3 p_3 - 380 p_1^4 p_3^2 + 2370 p_1^2 p_2 p_3^2 \\ & - 3630 p_2^2 p_3^2 + 10 p_1 p_3^3 - 310 p_1^6 p_4 + 2480 p_1^4 p_2 p_4 - 6270 p_1^2 p_2^2 p_4 \\ & + 4960 p_2^3 p_4 - 1030 p_1^3 p_3 p_4 + 2760 p_1 p_2 p_3 p_4 - 40 p_3^2 p_4 \\ & + 2040 p_1^2 p_4^2 - 5440 p_2 p_4^2, \end{aligned}$$

S_6 = is not required, because P_6 can be calculated in another manner.

Next, we have,

$$P_1 = S_1 = 3p_1^2 - 8p_2 = 48(b^2 - ac)/a^2.$$

$$\begin{aligned} P_2 = \tfrac{1}{2}(P_1 S_1 - S_2) = {} & 3p_1^4 - 16 p_1^2 p_2 + 22 p_1^2 - 2 p_1 p_3 + 8 p_4 \\ = {} & 8(96b^4 - 192ab^2 c + 99a^2 c^2 - 4a^2 bd + a^3 e)/a^4, \end{aligned}$$

$$\begin{aligned} P_3 = \tfrac{1}{3}(P_2 S_1 - P_1 S_2 + S_3) = {} & p_1^6 - 8p_1^4 p_2 + 24 p_1^2 p_2^2 - 28 p_2^3 - 8 p_1^3 p_3 \\ & + 30 p_1 p_2 p_3 - 26 p_3^2 + 6 p_1^2 p_4 - 16 p_2 p_4 = 32(128 b^6 - 384 ab^4 c \\ & + 432 a^2 b^2 c^2 - 189 a^3 c^3 - 64 a^2 b^3 d + 90 a^3 bcd - 13 a^4 d^2 + 3 a^3 b^2 e - 2 a^4 ce)/a^6, \end{aligned}$$

$$\begin{aligned} P_4 = \tfrac{1}{4}(P_3 S_1 - P_2 S_2 + P_1 S_3 - S_4) = {} & 2p_1^4 p_2^2 - 12 p_1^2 p_2^3 + 17 p_2^4 - 6 p_1^5 p_3 \\ & + 38 p_1^3 p_2 p_3 - 54 p_1 p_2^2 p_3 - 25 p_1^2 p_3^2 + 48 p_2 p_3^2 \\ & + 6 p_1^4 p_4 - 32 p_1^2 p_2 p_4 + 24 p_2^2 p_4 + 56 p_1 p_3 p_4 - 112 p_4^2 \\ = {} & 16(1152 b^4 c^2 - 2592 ab^2 c^3 + 1377 a^2 b^4 - 96 b^5 d + 3648 ab^3 cd \\ & - 1944 a^2 bc^2 d - 400 a^2 b^3 d^2 + 288 a^3 cd^2 + 96 ab^4 e - 192 a^2 b^2 ce \\ & + 54 a^3 c^2 e + 5 ba^3 bde - 7 a^4 e^2)/a^6, \end{aligned}$$

$$\begin{aligned} P_5 = \tfrac{1}{5}(P_4 S_1 - P_3 S_2 + P_2 S_3 - P_1 S_4 + S_5) \\ = {} & p_1^2 p_2^4 - 4 p_2^5 - 6 p_1^3 p_2^2 p_3 + 26 p_1 p_2^3 p_3 + 9 p_1^4 p_3^2 - 42 p_1^2 p_2 p_3^2 \\ & - 18 p_2^2 p_3^2 + 54 p_1 p_3^3 + 6 p_1^2 p_2^2 p_4 - 32 p_2^3 p_4 - 18 p_1^3 p_3 p_4 + 120 p_1 p_2 p_3 p_4 \\ & - 216 p_3^2 p_4 - 72 p_1^2 p_4^2 + 192 p_2 p_4^2 \\ = {} & 1152(18 b^2 c^4 - 27 ac^5 - 48 b^3 c^2 d + 78 abc^3 d + 72 b^4 d^2 - 56 ab^2 cd^2 \\ & - 9 a^2 c^2 d^2 + 12 a^2 bd^3 + 3 ab^2 c^2 e - 6 a^2 c^3 e - 4 ab^3 de + 10 a^2 bcde \\ & - 3 a^3 d^2 e - a^2 b^2 e^2 + a^4 ce)/a^6, \end{aligned}$$

$$P_6 = 256\left\{(ae - 4bd + 3c^2)^3 - 27(ad^2 + eb^2 + c^3 - ace - 2bcd)^2\right\}/a^6.$$

This last is from TODHUNTER, *Equations*, p. 177.

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13369. (Rev. J. CULLEN, B.A.)—Solve the partial differential equation

$$z\left\{\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2\right\}=x-y.$$

Solution by H. W. CURJEL, M.A.

CHARPIT'S auxiliary equations are

$$\frac{dp}{p\,(p^2-q^2)-1}=\frac{dq}{q\,(p^2-q^2)+1}=\frac{dz}{-2z\,(p^2-q^2)}=\&c.;$$

hence $2z\,(dp+dq)+(p+q)\,dz=0\;;\quad\therefore\;p+q=a/z^{\frac{1}{2}}.$

From this and the given equation

$$p=\frac{1}{2z^{\frac{1}{2}}}\left(a+\frac{x-y}{a}\right),\quad q=\frac{1}{2z^{\frac{1}{2}}}\left(a-\frac{x-y}{a}\right);$$

therefore, substituting in $p\,dx+q\,dy=dz$ and integrating, we get

$$4z^{\frac{1}{2}}=3a\,(x+y+b)+\frac{3}{2a}(x-y)^2.$$

This is the complete integral; the other integrals can be deduced in the usual way.

12866. (C. EWING.)—Find three cube numbers whose sum will be both a cube and a square.

Solution by MORGAN BRIERLEY.

By a known method (YOUNG'S *Algebra*, p. 322), find three cubes = a cube; thus $3^3+4^3+5^3=6^3$; then $3\,.\,6^3\,.\,\chi^2$, $4\,.\,6^3\,.\,\chi^2$, and $5\,.\,6^3\,.\,\chi^2$, are the roots of the cubes required. χ may be assumed at pleasure, say $\chi=\frac{1}{6}$; the three roots are 18, 24, and 30, and $18^3+24^3+30^3=36^3=216^2$.

Note on Question 13267 *by* R. CHARTRES.

As R might fall on EF produced, the proof given (in Vol. LXVII., pp. 33, 34) is not satisfactory.

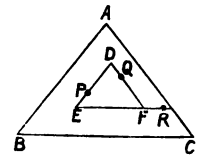
But the mean value of the part of PQR between AD and AE = $\frac{1}{3}$APE, and the mean value of

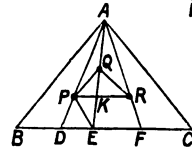
APE = $\frac{1}{2}$ADE, or $\frac{1}{8}\Delta$;

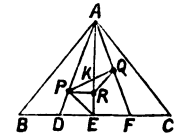
therefore mean value of the part of PQR between AD and AE = $\frac{1}{24}\Delta$, and similarly for the part between AE and AF.

Therefore mean value of

PQR = $\frac{1}{12}\Delta$.







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APPENDIX.

UNSOLVED QUESTIONS.

5171. (J. Dawson.)—Given the area and angles of a triangle, construct it by the First Book of Euclid; and bisect a given quadrilateral by a line parallel to one of its sides, within the same limits.

5174. (W. W. Johnson.)—It is stated that, if a knight be placed on any square of a chessboard, the knight's tour can be effected by simply observing the rule to play the piece to one of the squares from which he will command the smallest number of squares which he has not already occupied. Is this demonstrably true? When a tour is formed in this manner, the knight cannot, however, generally return to the original square on the 64th move, as is the case with the symmetrical tours.

5177. (S. Watson.)—Three radii of a given circle are drawn at random, and upon them three points are taken at random, one on each; find the average area of the triangle formed by joining the points.

5179. (Professor Crofton, F.R.S.)—A four-sided frame is found to be in equilibrium, with four given forces acting at the joints; though, if the joints were free, the equilibrium could not subsist; determine the *greatest* bending moment which can be in operation on any specified joint, that is, that which would exist there if the three remaining joints were free.

5187. (Editor.)—Through a given point draw a straight line that shall cut off from the sides of a given angle a triangle whose perimeter—or the perimeter with the base taken *negatively*—shall be given. Required a discussion of the various cases that may arise, and a solution both by geometry and by algebra.

5188. (Christine Ladd.)—Prove that the product of the lengths of the tangents, from the radical centre of three circles, to any pair of circles through the intersections of the given circles, is equal to the product of the lengths of the tangents, from the same point, to any pair of circles tangential to the given circles.

5191. (Rev. H. G. Day, M.A.)—In Question 5136, if n bricks are *supported* till the pile is raised, find the chance that it will stand.

[The question referred to (by Professor Hendricks, M.A.) is as follows:—If four bricks are placed on each other at random, with their longest axes horizontal and in the same vertical plane, show that the probability that the pile will stand is $\frac{209}{2304}$.]

5199. (R. Battle, M.A.)—Find a circular sector that is divided by its chord into parts having a given ratio.

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5201. (J. Dawson.)—Divide a given straight line, by the First Book of Euclid, into two parts such that the equilateral triangle on one part shall be n times that on the other part.

5202. (J. Richards, M.A.)—Place a chord in a given conic, so that the lines joining its ends with a given point may contain a given angle, and form with it the least possible triangle.

5205. (A. Martin.)—Four spheres, of radii a, b, c, d, touch each other externally, each touching the other three; four spheres, of radii w_1, x_1, y_1, z_1, are drawn in the space enclosed by them, each touching the other three and three of the given spheres; and four spheres, of radii w_2, x_2, y_2, z_2, are drawn in the space enclosed by the first group of inscribed spheres, each touching the other three and three of the first group of inscribed spheres, and so on; find the radii w_n, x_n, y_n, z_n, of the nth group of inscribed spheres.

5209. (Professor Cayley.)—In a hexahedron ABCDA′B′C′D′, the plane faces of which are ABCD, A′B′C′D′, A′ADD′, D′DCC′, C′CBB′, B′BAA′, the edges AA′, BB′, CC′, DD′ intersect in four points, say AA′, DD′ in α; BB′, CC′ in β; CC′, DD′ in γ; AA′, BB′ in δ: that is, starting with the duad of lines $\alpha\beta$, $\gamma\delta$, the four edges AA′, BB′, CC′, DD′ are the lines $\alpha\delta$, $\beta\delta$, $\beta\gamma$, $\alpha\gamma$, which join the extremities of these duads. Similarly the four edges AB, CD, A′B′, C′D′ are the lines joining the extremities of a duad; and the four edges AD, BC, A′D′, B′C′ are the lines joining the extremities of a duad. The question arises: "Given two duads, is it possible to place them in space so that the two tetrads of joining lines may be eight of the twelve edges of a hexahedron?"

[The duad $\alpha\beta$, $\gamma\delta$ is considered to be given when there is given the tetrahedron $\alpha\beta\gamma\delta$ which determines the relative position of the two finite lines $\alpha\beta$ and $\gamma\delta$.

5210. (Professor Clifford, F.R.S.)—In a polyhedron having n summits and only triangular faces (Δ-faced n-acron, Cayley), let every plane which contains three summits, but is not a face, be called a diagonal plane; and let every line which contains two summits, but is not an edge, be called a diagonal line: then (a) there is a surface of class $n-4$ touching all the diagonal planes; (b) this surface contains all the diagonal lines; (c) the conditions of passing through the diagonal lines and touching the diagonal planes are just sufficient to determine the surface, and no more; and (d) when the surface touches the plane at infinity the volume of the polyhedron is zero.

5214. (Professor Tait, M.A.)—Show that, whatever functions of x be represented by y and z, we have always $\dfrac{\int yz\,dx}{\int y\,dx} > \epsilon^{\int y \log z\,dx \,/ \int y\,dx}$, all the integrals being taken between the same limits of x; and all the quantities involved being positive.

5215. (Professor Salmon, F.R.S.)—Given five conics, it is of course possible in an infinity of ways to determine five constants a, b, c, &c., so that $aU_1 + bU_2 + cU_3 + dU_4 + eU_5$ may be either a perfect square L^2 or the product of two factors M, N. Prove that the lines L, M touch a conic, and that the lines M, N are conjugate with regard to that conic; from which it follows that, if M be given, N passes through a fixed point.

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5217. (Colonel Clarke, C.B., F.R.S.)—P and Q are points on a diameter, and equidistant from the centre of a given circle C. Another circle wholly within C encloses one of these points; find the chance that it encloses both.

5219. (Dr. Booth, F.R.S.)—Show that to inscribe the maximum parabola in a triangle involves the trisection of an angle.

5220. (W. S. B. Woolhouse, F.R.A.S.)—A solid and homogeneous hemisphere of cast iron, the radius of which is 12 inches, is placed with its vertex downwards upon a horizontal plane, and another, of the same description, of radius 8 inches, is similarly placed upon the former, with its vertex in contact with the centre of the base. Determine their simultaneous oscillations, supposing sliding to be prevented by the friction of the surfaces. Also suppose a slight vertical impact to be given to the upper hemisphere, and find the periods of the vibrations which will be communicated by it to the lower hemisphere, as well as the amount of friction at both surfaces. Again, such motions may be given as will cause the two hemispheres to oscillate freely in corresponding and equal times, either in the same or opposite directions; find the times of these oscillations, and the quantity of friction in both cases.

5221. (Elizabeth Blackwood.)—A random point A is taken within a given circle, and three random points P, Q, R are taken in the circumference; find the chance that AP, AQ, AR can be the sides of an acute-angled triangle.

5229. (R. F. Scott, M.A.)—Prove that

$$1+\tan\log x = \cfrac{x(1^2+1)}{3x-\cfrac{x^2(2^2+1)}{5x-\cfrac{x^2(3^2+1)}{7x-\cfrac{x^2(4^2+1)}{9x-\&c.}}}}.$$

5233. (W. H. H. Hudson, M.A.)—In a communistic society in which every one is equally frugal and industrious and devotes an amount (X) of his wealth to production, the wealth produced by each man's labour per unit of time is x times the capital with which he works. If the population increases at the rate of $100p$ per cent. per unit of time, prove that each man's wealth will continually increase so long as X bears a greater ratio to his whole wealth than $p : x$; X, x, p being all supposed to vary continuously.

5237. (W. S. Burnside, B.A.)—Determine a property of the surface which renders the double integral $\iint\left\{N\left(\frac{1}{R}+\frac{1}{R_1}\right)-\frac{P^2}{RR_1}\right\}dx\,dy$ a maximum; where N is the normal, P the perpendicular from the origin on the tangent plane, and R, R_1 the principal radii of curvature.

5239. (C. Taylor, M.A.)—The bisectors of the vertical angle A of a triangle meet the base in P, Q; trace the variations of magnitude in AO : OP and AO : OQ as O moves along the base, and apply the result to prove that a conic is concave to its axis.

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5252. (Editor.)—Find integral values of x, y, z that will satisfy each of the following equations separately :—

$$y^2-4x^2 = 2z, \quad y^2-4x^2 = 5z, \quad y^2-6x^2 = 10z, \quad y^2-7x^2 = 48z.$$

5253. (Elizabeth Blackwood.)—C is the centre of a given circle, X, Y are random points in the area, and P is any point in the circumference; two concentric circles, with radii CX, CY, divide CP into three segments: find the chance that these three segments can be the sides of an acute-angled triangle.

5266. (J. J. Walker, M.A.)—If the line which is the locus of the equation $lx + my + nz = 0$ meet the sides BC, CA, AB of the triangle of reference in the points A′, B′, C′, and the bisectors of the angles A, B, C in the points D, E, F respectively, prove that

$$m : n = \frac{C'A'}{C'D} : \frac{B'A'}{B'D}, \quad n : l = \frac{A'B'}{A'E} : \frac{C'B'}{C'E}, \quad l : m = \frac{B'C'}{B'F} : \frac{A'C'}{A'F}.$$

5284. (J. J. Walker, M.A.)—Prove that the discriminant of the quartic

$$ax^4 + by^4 + cz^4 + 4dx^2y + 4ex^2z$$

is

$$\{27b^2c^2(be^4 + cd^4) - a^3b^3c^3\}^3 - (27)^3a^3b^8c^8d^4e^4.$$

5285. (E. B. Elliott, M.A.)—S, U, V, W are four fixed quadrics having a common plane section; through any point P are drawn six other quadrics having the same common plane section, and each passing also through the second plane section common to some two of S, U, V, W. Show that these six have one more point common. Show also that, for different positions of P lying in any fixed plane, the locus of this second common point is in general a quadric through the plane section common to S, U, V, W.

5290. (W. H. H. Hudson, M.A.)—Trace the curve $r(c - r\sin\theta) = a^2$; showing (1) the change of form of the curve as a increases from 0 to a large value; (2) distinguishing clearly the parts of the curve that arise as θ passes from 0, through each quadrant, up to four right angles; (3) when $a > \frac{1}{2}c$, find the tangents that pass through the origin; and, (4) when $a = \frac{1}{2}c$, prove that the distance of the points of inflexion from the origin is $3a$.

5291. (Hugh McColl, B.A.)—If m points be taken at random in a given magnitude, and the magnitude is then divided into n random parts; show that the chance that some one of the n parts contains all the m points is

$$\frac{m!\,n!}{(m+n-1)!}.$$

[Hence Professor Crofton's theorem in Question 5213 may be proved, and extended to the tetrahedron without the aid of the Integral Calculus. Question 5213 is as follows:—If three points X, Y, Z are taken at random within a triangle ABC, show (as may be done without integration) that it is an even chance that both Y and X lie on some one of the triangles AXB, AXC, BXC.]

5311. (Christine Ladd.)—If T_1, T_2, T_3 be the squares of the respective distances from the centres of the escribed circles to the orthocentre of a triangle, and D_1, D_2, D_3 the squares of the distances from the same points

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to the centre of the circumscribed circle, show that

$$\begin{vmatrix} 1 & 1 & 1 \\ T_1 & T_2 & T_3 \\ D_1 & D_2 & D_3 \end{vmatrix} = 0.$$

5318. (Hugh McColl, B.A.)—If n points are taken at random in a given magnitude, and the magnitude be then divided into n random parts, show (without using the Integral Calculus) that the chance that each part contains exactly one point is $\frac{n!\,(n-1)!}{(2n-1)!}$.

[A solution of this and of Question 5291 will involve a solution of Professor Crofton's Question 5273. Both of these are given above, under 5291.]

5326. (R. Battle, M.A.)—Find the complete primitive of the equation

$$\frac{d^5y}{dx^5} + \frac{d^3y}{dx^3} + \frac{dy}{dx} = \frac{d^4y}{dx^4} \cdot \frac{d^2y}{dx^2},$$

and the relation between x and y in

$$x^{11}y^5\frac{dy}{dx} + x^{16}y^{12} = a.$$

5328. (Professor Cayley.) — Mention what form of given relation $\phi\,(a, b, c, \ldots) = 0$ between the roots of a given equation will in general serve for the rational determination of the roots; explain the case of failure; and state what information as to the roots is furnished by a given relation not of the form in question.

5332. (Editor.) — Through a given point S a variable straight line PSQ is drawn, cutting a given circle in the points P and Q, which are joined to another given point H; find (1) the average area of the triangle PHQ, and (2) the greatest triangle of the series.

5358. (Professor Cayley, F.R.S.)—Find at any point of a plane curve the angle between the normal and the line drawn from the point to the centre of the chord parallel and indefinitely near to the tangent at the point; and examine whether a like question applies to a point on a surface and the indicatrix section at such point.

5360. (Professor Crofton, F.R.S.)—An hexagonal frame is stiffened by three diagonal braces, joining each pair of opposite joints. The condition that it shall be possible for the frame to be *self-strained*, that is, in a state of internal stress, without the action of any external forces is either (1) that the six joints shall lie on a conic section; or (2) that the three diagonals shall meet in a point, that is, that the hexagon shall be circumscribed about a conic section. A frame of less than six joints, containing only the number $(2n-3)$ of bars requisite for stiffness, cannot be self-strained (unless three joints lie in a straight line).

5361. (Professor Wolstenholme, M.A.)—If a heavy particle be projected so as to move on the inside of a smooth paraboloid of revolution, whose axis is vertical and vertex downwards, the pressure on the surface at any point P varies as the curvature of the meridian at that point, and the resolved vertical pressure is to the weight of the body as SL . SM : SP^2, where L, M are the highest and lowest points of the path and S the focus.

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5365. (Christine Ladd.)—If P be the orthocentre of a triangle, I the centre of the inscribed circle, and Q the centre of the circle inscribed in the orthocentric triangle (that is, the triangle formed by joining the middle points of the sides), prove that $QP^2 - QI^2 = 2(Rr - R\rho - r^2)$.

5366. (Editor.)—Find the locus of a point whose distance from one of three given points is (1) an arithmetic mean, (2) a geometric mean, between its distances from the other two.

5383. (A. Martin.)—From a given point in the circumference of a circle a chord is drawn in a random direction, and then two points are taken on it at random, and a third point anywhere in the surface of the circle; find the chance that the triangle formed by joining these points is acute.

5385. (J. Dawson)—Prove, by the First Book of Euclid, that the middle points of a complete quadrilateral are in a straight line.

5401. (Col. A. R. Clarke, C.B., F.R.S.)—Determine the positions in which a cube would float.

5408. (Rev. W. Roberts, M.A.)—Let the base AB of a spherical triangle be given, and consider the curve which is the locus of the vertex C, when

$$\cos AB \sin^{\frac{1}{2}} AC - \cos BC \sin^{\frac{1}{2}} BC = (\sin AC - \sin BC)(k^2 - \sin AC - \sin BC)^{\frac{1}{2}},$$

k being a constant quantity. Prove that the system of curves obtained by supposing k to vary may be found by writing in the above equation the angle BAC instead of AC, and the supplement of ABC instead of BC.

5414. (S. Watson.)—On the sides of a regular polygon of n sides, n points are taken at random, one on each, which points being joined, an inscribed polygon of n sides will be formed. Again, inside the n triangles outside this last polygon, n points are taken at random, one in each, forming, when joined, another polygon of n sides, the average area of which it is required to find.

5418. (J. Dawson.)—Show that a triangle, having a given angle, and one of the sides about that angle, n times the other, may be inscribed in a given triangle, by the First Book of Euclid.

5423. (Professor Clifford, F.R.S.)—It is known that, if four lines be given, the circles circumscribing the four triangles so formed meet in a point; and that, if five lines be given, the five points so belonging to their five tetragrams lie on a circle (Miquel's Theorem, see *Diary* for 1861, p. 55). Show that this series of propositions is interminable, so that, if $2n$ lines be given, they determine $2n$ circles that meet in a point; and, if $2n+1$ lines be given, they determine $2n+1$ points that lie on a circle.

5424. (Professor Crofton, F.R.S.)—Prove that the mean value of the triangle formed by joining the centroid of a triangle and two points taken at random within it is $\frac{\Delta}{3^6}(34 + \frac{16}{3}\log 2)$.

5425. (Professor Chase, LL.D.)—If f represent any force whatever that varies inversely as the square of the distance, and r be the radius of a perpetual circular oscillation produced by the force, find the mean velocity of a synchronous radial oscillation.

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5451. (T. Cotterill, M.A.) — In the resultant of a quadratic $Ax^2 + Bxy + Cy^2$, and the general form $\phi(x, y)$, the part independent of B is $\phi(x, y) . \phi(x, -y)$, in which $x^2 = +C$ and $y^2 = -A$, and consequently the sum or difference of two real squares according as A and C have or have not the same sign. The parts independent of A and C are, respectively, $\phi(1, 0) . \phi(C, -B)$ and $\phi(0, 1) . \phi(-B, A)$.

5453. (Professor Cayley, F.R.S.)—If B, C, Q, R are any four given points, find the locus of a fifth point A, such that

$$AQ . AR . BC + BQ . BR . CA + CQ . CR . AB = BC . CA . AB.$$

[The construction of one point on the locus is given by the theorem in Question 2010, *Reprint*, Vol. VII., p. 43, which was proposed by the Editor, who required proof of an equation identical with the above, supposing Q, R to be the foci of an ellipse inscribed in a triangle ABC.]

5455. (Professor Crofton, F.R.S.)—Prove that the mean area of the triangle formed by three points taken at random within any plane convex area, Ω, is

$$M = \Omega - \frac{1}{\Omega^3} \iint C^3 S^2 \, dp \, d\omega,$$

where C is the length of any chord crossing the area; p, ω, the *coordinates* of the line C, referred to any pole O inside the area and any fixed axis through O, and S the area of either segment cut off from the area Ω by the chord C; the integration extending to all possible positions of the chord.

5457. (Professor Clifford, F.R.S.)—A triangle ABC has its vertices A, B jointed to two rods AD, BE, which can turn about the fixed points D, E; express the coordinates of the point C in terms of elliptic functions of a single parameter.

5465. (Elizabeth Blackwood.)—If X, Y are two random points within a circle, and P a point in a fixed diameter equidistant from X and Y, show that the chance that P is nearer to X (or Y) than it is to the circumference is $\frac{2}{5}$; and hence solve the Editor's Question 1843, namely:—"Three points being taken at random inside a circle, show that the chance that the circle drawn through them will lie wholly within the given circle is $\frac{1}{4}$."

[A solution of Question 1843 is given on pp. 95–99 of Vol. XIII. of the *Reprint*. See also the solution of the Editor's Question 3335, on pp. 80, 81 of Vol. XV. of the *Reprint*.]

5471. (H. Cookson, M.A.)—A man can read with the naked eye the small print of a placard at a distance of 11 inches, and with the aid of spectacles he can read the same print with ease at a distance of 7 feet 7 inches; find the focal length of the spectacles used.

5476. (Colonel A. R. Clarke, C.B., F.R.S.)—Three points are taken at random, one on each of three faces of a tetrahedron. Find the chance that the plane passing through them cuts the fourth face.

5477. (R. W. Genese, M.A.)—A stick has a uniform heavy string suspended from its two ends. If the stick be made to turn round any point, find the locus of the vertex and the envelopes of the string and its directrix.

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5486. (Dr. Traill.)—Let $\nu_1 = a$ be the hyperboloid which gives a line of curvature on the ellipsoid (ρ). Take any geodesic touching this line of curvature, and let μ, ν be the two confocal hyperboloids at any point on this geodesic. Draw any common tangent line to these two hyperboloids, and through this line draw a tangent plane to the hyperboloid $\nu_1 = a$. Prove that the angle between this plane and the tangent plane to the hyperboloid (ν) at the point where the common tangent touches it is equal to the angle between the geodesic and the line of curvature.

5495. (Professor Kelland, F.R.S.)—Find, by quaternions, the locus of the intersections of perpendiculars from a given point on planes which pass through a given point.

5500. (Professor Evans, M.A.) Construct a triangle, having given the product of two sides, the medial line to the third side, and the difference of the angles adjacent to the third side.

5505. (Christine Ladd.)—If, from two points on the directrix which subtend a constant angle at the focus, tangents to a conic be drawn, they will intersect on another conic having the same focus and directrix.

5509. (J. J. Walker, M.A.)—If u is any ternary quartic form, show that the result of eliminating k from the conditions that

$$\lambda^3\left(\frac{d}{dx}+k\frac{d}{dy}\right)^4 u+\lambda^2\mu\left(\frac{d}{dx}+k\frac{d}{dy}\right)^3 u+\lambda\mu^2\left(\frac{d}{dx}+k\frac{d}{dy}\right)^2 u+\mu^3\left(\frac{d}{dx}+k\frac{d}{dy}\right)u$$

shall be a perfect cube is unaltered by substituting $\frac{d}{dx}$ for $\frac{d}{dy}$ in the above cubic in $\lambda\mu$.

5513. (C. Leudesdorf, M.A.)—A bag contains $4n$ balls, each of which is equally likely to be red, black, white, or yellow. Of these, all but four are drawn out at random, and are found to consist of $n-1$ of each colour. Find the chance that the remaining four may be (i.) one of each colour, (ii.) all of one colour.

5525. (Professor Minchin, M.A.)—1. A lamina is projected in its own plane. Prove that the space centroid is a straight line, and the centroid of the body is a circle.

2. A body, rotating about an axis through its centre of gravity, falls vertically. The space centroid is a parabola, and the centroid of the body is a spiral of Archimedes.

3. Given the motion of one point in a body, and also the space centroid, find the centroid of the body.

[We may remark here, as a bit of mathematical history, that the very convenient term *centroid* was, in reality, invented by the late T. S. Davies, and used by him first in Vol. XVI. of the *Edinburgh Transactions*, and afterwards in an article on *Radical Axes and Poles of Similitude* in the *Lady's and Gentleman's Diary* for 1851. We have persistently tried to get our correspondents to use the word *centroid* in place of *centre of gravity*, and, now that the word is finding its way into books, there is, we hope, some probability that it will altogether displace *centre of gravity*.]

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